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Dedication

To my Cailey, continue looking up to the sky at the stars, and you can reach them! Love, Mommy

To my family, thank you for your time and support. It wouldn’t have been possible without you!

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Christina Pawlowski-Polanish has taught all levels of math, including 12 years of AP Calculus BC at Commack High School in Commack, NY, and College Precalculus for 8 years. She earned a B.A. in mathematics and secondary education and an M.S. in mathematics from Hofstra University. She frequently presents on mathematics literacy at local, state, and national conferences and leads workshops in her district and at colleges. Christina is a member of many organizations, including the National Council of Teachers of Mathematics, Suffolk County Math Teachers' Association, and the Association of Mathematics Teachers of New York State, where she served as recording secretary and is presently a county chair. Christina was accepted into the New York State (NYS) Master Teacher Program and the NYS Academy of Master Teachers. She was chosen as Commack School District's Teacher of the Year and was a finalist for the 2018 NYS Teacher of the Year award. Christina joined the National Network of State Teachers of the Year and has been published on their blog. She was honored by the governor as a recipient of the NYS Empire State Excellence in Teaching Award in 2018. Most recently, Christina was named the 2019 New York State winner of the Presidential Award for Excellence in Mathematics and Science Teaching and was honored in Washington, D.C.

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Visit Barron's Online Learning Hub for an additional full-length practice test.

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How to Use This Book

This book provides comprehensive review and extensive practice for AP Precalculus. It is based on the latest AP Precalculus Course and Exam Description published by the College Board.

Introduction

Start with the Introduction, which contains an overview of the AP Precalculus course and exam. Learn the topics covered within each unit, familiarize yourself with the exam format, and review the guidelines for calculator use on test day. Then read through the Prior Knowledge section, which covers important tools and skills (that you may have learned in previous math classes) that can be applied to solve AP Precalculus problems.

Review and Practice

Study Chapters 1 through 12, which are organized according to the units of AP Precalculus. Every chapter includes Learning Objectives that will be covered, a review of each topic, dozens of tables and figures that illustrate key concepts, numerous examples with clear solutions, and multiple-choice practice questions (with detailed answer explanations) to check your progress. You'll also find helpful tips and advice to keep in mind for test day.

Note: The last three review chapters in this book focus on Unit 4 topics. While Unit 4 topics will not be tested on the AP Precalculus exam, they may be part of your school's course curriculum. Furthermore, reviewing

these chapters may help build your understanding of topics that you will encounter on the exam.

Practice Tests

This book concludes with two full-length practice tests that mirror the actual exam in format, content, and level of difficulty. Each test is followed by detailed answer explanations for all questions.

Glossary of Key Terms

Consult the Glossary of Key Terms at the end of this book. AP Precalculus is a vocabulary-based subject, so a strong command of the most frequently used precalculus terms will further prepare you for success on the exam.

Online Practice

There is also an additional full-length practice test online. You may take this test in practice (untimed) mode or in timed mode. All questions are answered and explained.

For Students

Students who use this book independently will improve their performance by studying the examples carefully and trying to complete the practice problems before referring to the answer explanations. You are strongly encouraged to read the book from start to finish since each topic builds on the previous ones. However, it is possible to use this book as a reference for specific topics that need review.

For Teachers

Teachers who use this book with a class can do so in several ways. You can assign all or part of each set of multiple-choice questions after the topic has been covered during the school year. These sets can also be used for review

purposes shortly before exam time. The practice tests will also be helpful in reviewing toward the end of the year.

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Barron's 5 Essential

As you review the content in this book to work toward earning that **5** on your AP PRECALCULUS exam, here are five things that you **MUST** know above everything else:

Develop an understanding of the content; don't just memorize. It is much better to understand the different mathematical concepts rather than memorizing properties and formulas. By understanding why the properties and formulas work, you will have a firm grasp of the topic, which will enable you to solve the problems. No two math questions are alike, and so it is impossible to memorize your way through a math course, especially an AP math course. This book often goes through the derivations of formulas and gives background explanations of the concepts so that you will understand how the math works.

Build your vocabulary. Math is its own language with its own notations and terms. Often terms in math can have multiple meanings within the context of the problem. For example:

- In trigonometry, *tangent* refers to the ratio of sides of a right triangle or the ratio of the trigonometric functions, $\frac{\sin\theta}{\cos\theta}$.

- In geometry, *tangent* refers to a line that intersects a circle in only one point.
- In calculus, *tangent* refers to a line that intersects a curve at a point, matching the curve's slope there.

Make sure you understand the specific mathematical meanings of each term, especially those in the Glossary at the end of this book. Most of the multiple-choice questions on the exam measure your knowledge and ability to apply mathematical vocabulary.

Mark your model. Since math is its own language, the way you read and approach a problem should be different from how you read in other subject areas. Many problems in math require visualization. If a model is provided, it is important to mark the model with any given information. The markings in the model may tell you about a relationship in the figure that will lead to an equation. If a verbal description is provided, begin by creating a model that can be marked.

Pause for punctuation. When you are reading a problem and see a comma, period, semicolon, colon, or the words *and* or *is*, you should stop and ask yourself, "Why might this break in the problem be important?" The answer to this question may tell you about some equation to focus on, information that will allow you to create an equation, or some important information that will lead to a solution to the problem. Think of these pauses as a way for the question writer to signal to you a list of important givens in an organized manner.

Improve your study habits. Reaching this level in math means that you have been a student for many years and are not afraid to take on a challenging course. You might already have your own study strategies in place or may have found math very easy up until now. Studying for the AP Precalculus exam is an opportunity for you to enhance your study strategies. Study strategies can include:

- Spacing your studying over a span of weeks as opposed to cramming the days before an exam.
- Reflecting on topics you struggled with, reading the review material on that section, and answering the practice questions. Repeating a question is a great learning tool to ensure you understand the concept.
- Using the practice questions at the end of each chapter to test your knowledge.
- Taking all of the practice tests to become familiar with the testing format and using them to understand which topics you need to repeat and practice more.
- Using flashcards to review important vocabulary or equations that need to be better understood. The terms you select to make cards on should be ones that are seen often or ones that you are not familiar with.

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Introduction

AP Precalculus is a full-year course designed to prepare you for the math you will encounter at the college level. This course should be taken after you have completed Geometry and Algebra 2 or once you have completed Integrated Math 3. AP Precalculus contains content that is already present in standard high school precalculus courses. This course will develop your ability to express functions, concepts, problems, and conclusions analytically, graphically, numerically, and verbally and to understand how these are related. You will also learn how to use a graphing calculator as a tool for mathematical investigations and problem solving.

Unit Outline

The course is divided into four units. The first three units are assessed on the AP Precalculus exam. The fourth unit includes additional topics available to schools but those topics are **not** included on the AP Precalculus exam.

Unit 1: Polynomial and Rational Functions

Unit 1 covers the following topics:

- Rates of change
- Linear functions
- Quadratic functions
- Complex zeros of polynomial functions
- End behavior of polynomial functions
- Zeros of rational functions

- Vertical asymptotes of rational functions
- Holes of rational functions, long division
- Transformations of functions, modeling

Unit 2: Exponential and Logarithmic Functions

Unit 2 covers the following topics:

- Arithmetic and geometric sequences
- Exponential functions
- Exponential function manipulation
- Data modeling
- Composition of functions
- Inverse functions
- Logarithmic expressions
- Inverses of exponential functions
- Logarithmic functions
- Logarithmic function manipulation
- Exponential and logarithmic equations
- Exponential and logarithmic inequalities, semi-log plots

Unit 3: Trigonometric and Polar Functions

Unit 3 covers the following topics:

- Periodic phenomena
- Sine function
- Cosine function
- Tangent function
- Graphs of trigonometric functions
- Transformations of sinusoidal functions
- Reciprocal trigonometric functions
- Solving trigonometric equations
- Trigonometric identity proofs

- Trigonometry and polar coordinates
- Polar function graphs

Unit 4: Functions Involving Parameters, Vectors, and Matrices

Unit 4 covers the following topics:

- Parametric functions
- Rectilinear motion
- Implicit functions
- Conic sections, vectors
- Vector operations
- Vector-valued functions
- Matrices, matrix operations
- Linear transformations and matrices
- Matrices as functions
- Markov chains

> These Unit 4 topics are **not** assessed on the AP exam.

Exam Format

The AP Precalculus exam is 3 hours long and is comprised of the following sections.

Section and Question Type		Number of Questions	Time	Exam Weighting
I (Multiple-Choice)	Part A—No Calculator	28	80 min	43.75%
	Part B—Calculator Active*	12	40 min	18.75%
II (Free-Response)	Part A—Calculator Active**	2	30 min	18.75%
	Part B—No Calculator	2	30 min	18.75%

*Some of these questions require the use of a graphing calculator.
**Some of these questions require the use of a graphing calculator. Note that, after 30 minutes, *you will no longer be permitted to use a calculator.*

Scoring of the Exam

Each completed AP exam receives a score according to the following five-point scale:

AP Score	Qualification
5	Extremely well qualified
4	Well qualified
3	Qualified
2	Possibly qualified
1	No recommendation

Many colleges and universities accept a score of 3 or better for college credit or advanced placement. Be sure to check the AP credit policies on individual colleges' websites.

Using Your Graphing Calculator on the AP Exam

Each student is expected to bring a graphing calculator to the AP exam. Different models of calculators vary in their features and capabilities; however, there are three procedures you must be able to perform on your calculator:

1. Produce the graph of a function within an arbitrary viewing window.
2. Solve an equation numerically.
3. Compute regression models numerically.

Guidelines for Calculator Use

1. For the multiple-choice questions in Section I: Part B, you may use any feature or program on your calculator.
2. On the free-response questions in Section II: Part A, remember the following:
    - You may use the calculator to perform any of the three listed procedures. When you do, you need to write only the equation or setup that will produce the solution and then the calculator result to the required degree of accuracy (three places after the decimal point unless otherwise specified).
    - For a solution for which you use a calculator capability other than the three listed above, you must write down the mathematical steps that yield the answer. A correct answer alone will not earn full credit.
    - When asked to "justify," you must provide mathematical reasoning to support your answer. Calculator results alone will not be sufficient.

WARNING

Don’t rely on your graphing calculator too much. Only a few of these questions require a calculator, and in some cases, it may be an inefficient use of time.

Additional Calculator Notes

- Learn the proper syntax for your calculator: the correct way to enter operations, functions, and other commands. Parentheses, commas, variables, or parameters that are missing or entered in the wrong order can produce error messages or yield wrong answers.
- Keep your calculator set in radian mode. Almost all questions about angles and trigonometric functions use radians. If you ever need to change to degrees for a specific calculation, return the calculator to radian mode as soon as that calculation is complete.
- Many calculators do not have keys for the secant, cosecant, or cotangent functions. To obtain these functions, use their reciprocals. For example, $\sec\left(\frac{\pi}{8}\right) = \frac{1}{\cos\left(\frac{\pi}{8}\right)}$.
- Evaluate inverse functions such as arcsin, arccos, and arctan on your calculator. Those function keys are usually denoted as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$.
- To achieve three-place accuracy in a final answer, do *not* round off variables at intermediate steps since this is likely to produce error accumulations. Round off only after your calculator produces the final answer.
- Although final answers expressed in this book are in simplest form, this is not necessary on Section II questions of the AP exam. According to the directions printed on the exam, “unless otherwise specified” (1) you need not simplify algebraic or numerical answers, and (2) answers that include decimals should be correct to three places after the decimal point. However, be aware that if you try to simplify, you must do so correctly or you will lose credit.

Prior Knowledge

Math is very much like a staircase; each course is a stair you climb, providing you with the skills and techniques to guide you upward to the next course. This section will give a brief review of important tools that are used to help solve AP Precalculus questions.

Factoring

Factoring is the process of breaking down a polynomial expression or equation and finding its factors that, when multiplied together, return to the original polynomial. A few different factoring techniques will be reviewed.

Greatest Common Factor (GCF)

Greatest common factor factoring refers to dividing out the greatest like term in a polynomial. The greatest common factor can be a number, a variable, or both.

› Example

Factor each of the following polynomials.

1. $3x^4 - x$
2. $2x^3 + 6x^4 - 4x^5$
3. $(x + 1)^3 - 4(x + 1)^2$

✓ Solution

1. There are two terms in this polynomial, $3x^4$ and x. Their greatest common factor is x. Factoring out x gives $3x^4 - x = x(3x^3 - 1)$.
2. There are three terms in this polynomial, $2x^3$, $6x^4$, and $4x^5$. Their greatest common factor is $2x^3$. Factoring out $2x^3$ gives $2x^3 + 6x^4 - 4x^5 = 2x^3(1 + 3x - 2x^2)$.
3. There are two terms in this polynomial, $(x + 1)^3$ and $4(x + 1)^2$. Their greatest common factor is $(x + 1)^2$. Factoring out $(x + 1)^2$ gives $(x + 1)^3 - 4(x + 1)^2 = (x + 1)^2((x + 1) - 4) = (x + 1)^2(x - 3)$.

Difference of Two Squares

Difference of two squares factoring refers to binomials of the form $x^2 - p^2$. To factor a difference of two squares, $x^2 - p^2 = (x - p)(x + p)$. The factors are known as conjugate pairs because each factor has the same terms, x and p, but one factor is a sum and the other factor is a difference.

› Example

Factor each of the following polynomials.

1. $16x^2 - 9$
2. $(y - 2)^2 - 25$
3. $49 - 4t^2$
4. $x^2 - 5$

✓ Solution

1. Two terms are being subtracted, $16x^2$ and 9. Factoring using the difference of two squares gives $16x^2 - 9 = (4x - 3)(4x + 3)$.
2. Two terms are being subtracted, $(y - 2)^2$ and 25. Factoring using the difference of two squares gives $(y - 2)^2 - 25 = ((y - 2) - 5)((y - 2) + 5) = (y - 7)(y + 3)$.
3. Two terms are being subtracted, 49 and $4t^2$. Factoring using the difference of two squares gives $49 - 4t^2 = (7 - 2t)(7 + 2t)$.
4. Two terms are being subtracted, x^2 and 5. Factoring using the difference of two squares gives $x^2 - 5 = (x - \sqrt{5})(x + \sqrt{5})$.

Factoring Trinomials

To factor a quadratic of the form $x^2 + bx + c$, look for two real numbers, p and q, that satisfy $p + q = b$ and $pq = c$.

› Example

Factor each of the following polynomials.

1. $y^2 - 24y + 144$
2. $x^2 + 6x + 8$
3. $z^2 - z - 20$

☑ Solution

1. Two numbers that multiply to $c = 144$ are -12 and -12, and these same two numbers add to $b = -24$. Therefore, $y^2 - 24y + 144 = (y - 12)(y - 12)$.
2. Two numbers that multiply to $c = 8$ are 4 and 2, and these same two numbers add to $b = 6$. Therefore, $x^2 + 6x + 8 = (x + 4)(x + 2)$.
3. Two numbers that multiply to $c = -20$ are -5 and 4, and these same two numbers add to $b = -1$. Therefore, $z^2 - z - 20 = (z - 5)(z + 4)$.

To factor a quadratic of the form $ax^2 + bx + c$ where $a \neq 0$ and $a \neq 1$, you can either guess and check or use the split the middle method.

Steps for Factoring Using Split the Middle

1. Find the product ac.
2. Determine numbers whose product is ac and whose sum is b.
3. Split up the middle term bx, and write it as a sum of the two numbers determined in step 2.
4. The trinomial is written as an equivalent polynomial with four terms. Group together the first two terms, group together the last two terms, and factor out a common factor from each.
5. A common factor for the resulting two terms should appear that can then be factored out. Doing this will reveal the two factors of the original trinomial.

› Example

Factor each of the following quadratics.

1. $25t^2 + 10t + 1$
2. $3a^2 - 10a + 8$

✓ Solution

1. Since this trinomial has an a-value other than 1, factor using the split the middle method. The product of a and c is 25. Two numbers that multiply to 25 are 5 and 5, and they also add to the b-value of 10.
   Split the middle term:

   $$25t^2 + 5t + 5t + 1$$

   Factor by grouping:

   $$(25t^2 + 5t) + (5t + 1)$$

   Take out a common factor from each set of parentheses:

   $$5t(5t + 1) + 1(5t + 1)$$

   Factor out the common factor $(5t + 1)$ from each term:

   $$(5t + 1)(5t + 1)$$

2. Since this trinomial has an a-value other than 1, factor using split the middle. The product of a and c is 24. Two numbers that multiply to 24 are -6 and -4, and they also add to the b-value of -10.
   Split the middle term:

   $$3a^2 - 6a - 4a + 8$$

   Factor by grouping:

   $$(3a^2 - 6a) + (-4a + 8)$$

   Take out a common factor from each set of parentheses:

$$3a(a-2)-4(a-2)$$

Factor out the common factor $(a-2)$ from each term:

$$(3a-4)(a-2)$$

Sum or Difference of Cubes

To factor a sum or difference of cubes, use the factoring formulas below:

$$x^3+p^3=(x+p)(x^2-px+p^2)$$
$$x^3-p^3=(x-p)(x^2+px+p^2)$$

The trinomial obtained by factoring a sum or difference of cubes does not factor further.

› Example

Factor each of the following polynomials.

1. x^3-8
2. $1+1000x^3$

✓ Solution

1. The polynomial is an example of a difference of cubes where $p=2$ since $2^3=8$. Use the factoring formula:

$$x^3-8=(x-2)(x^2+2x+2^2)=(x-2)(x^2+2x+4)$$

2. The polynomial is an example of a sum of cubes where $x=1$ since $1^3=1$ and $p=10x$ since $(10x)^3=1000x^3$. Use the factoring formula:

$$1+1000x^3=(1+10x)(1^2-10x\cdot 1+(10x)^2)=(1+10x)(1-10x+100x^2)$$

The Quadratic Formula

When asked to solve a quadratic equation, one method is to use factoring and set each factor equal to zero. Sometimes it is impossible to factor a

quadratic equation. In those cases, the solutions can be found using the quadratic formula.

The Quadratic Formula

Given a quadratic equation in standard form, $ax^2 + bx + c = 0$, the formula below can find the solutions to the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

› Example

Solve the following quadratic equations, and write your answer in simplest radical form.

1. $2x^2 - 5x - 7 = 0$
2. $4x^2 = 16x + 20$

✓ Solution

1. The quadratic equation is in standard form with $a = 2$, $b = -5$, and $c = -7$. Substitute the values into the quadratic formula:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-7)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{25 + 56}}{4}$$

$$x = \frac{5 \pm \sqrt{81}}{4}$$

$$x = \frac{5 \pm 9}{4}$$

$$x = \frac{5 + 9}{4}, x = \frac{5 - 9}{4}$$

The solutions are $x = \frac{7}{2}$, $x = -1$.

2. The quadratic equation needs to be written in standard form first, $-4x^2 + 16x + 20 = 0$. The equation is now in standard form with $a = -4$, $b = 16$, $c = 20$. Substitute the values into the quadratic formula:

$$x = \frac{-(16) \pm \sqrt{(16)^2 - 4(-4)(20)}}{2(-4)}$$

$$x = \frac{-16 \pm \sqrt{256 + 320}}{-8}$$

$$x = \frac{-16 \pm \sqrt{576}}{-8}$$

$$x = \frac{-16 \pm 24}{-8}$$

$$x = \frac{-16 + 24}{-8}, x = \frac{-16 - 24}{-8}$$

The solutions are $x = -1$, $x = 5$.

Imaginary Numbers

There are many different sets of numbers, such as integers, natural numbers, rational numbers, and real numbers. There is also the set of imaginary numbers. This occurs when there is a negative number under a square root.

The most basic imaginary number is $\sqrt{-1}$ and is notated as i. In other words, $i = \sqrt{-1}$. Operations using imaginary numbers are much like operations in the real number system.

$$\text{Addition: } 2i + 3i = 5i$$

$$\text{Subtraction: } 7i - 3i = 4i$$

When multiplying imaginary numbers, the exponents of the imaginary number are added together and their coefficients are multiplied.

$$\text{Multiplication: } 2i \cdot 5i = 10i^2$$

Special properties are associated with the powers of i, as noted in the sidebar to the right.

Powers of *i*

$$i = \sqrt{-1}$$
$$i^2 = (\sqrt{-1})^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$$

The powers continue to cycle following this pattern.

When dividing imaginary numbers, the exponents of the imaginary numbers are subtracted and their coefficients are divided.

$$\text{Division: } \frac{49i^5}{7i^3} = 7i^2 = 7(-1) = -7$$

Imaginary numbers can appear when working with square roots. A problem may ask for a solution to be written in simplest $a + bi$ form, which is a complex number. The a represents the real portion of the number, and the bi represents the imaginary portion of the number.

To simplify a complex number, first rewrite all radicals with a negative radicand as imaginary numbers by removing the negative sign and replacing it with the imaginary number i. Finally, simplify the radical.

› Example

Solve the quadratic equation, and write your solution in simplest $a + bi$ form.

$$x^2 + 6x + 13 = 0$$

✓ Solution

The quadratic equation is written in standard form with $a = 1$, $b = 6$, $c = 13$. Use the quadratic formula:

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(13)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{36 - 52}}{2}$$

$$x = \frac{-6 \pm \sqrt{-16}}{2}$$

$$x = \frac{-6 \pm i\sqrt{16}}{2}$$

$$x = \frac{-6 \pm 4i}{2}$$

$$x = \frac{-6 + 4i}{2}, x = \frac{-6 - 4i}{2}$$

The solutions are $x = -3 + 2i$, $x = -3 - 2i$.

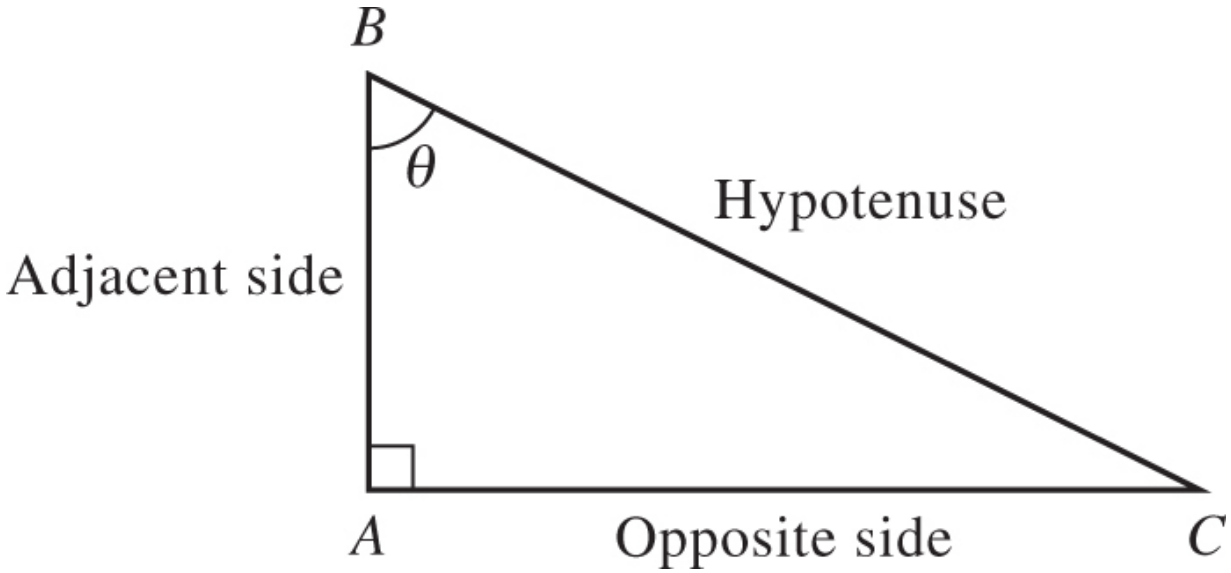
Right Triangle Trigonometry

Right triangles are made up of a right angle and two acute angles. The sum of the 3 angles is 180°. The sides opposite the acute angles are called legs, and the side opposite the right angle is called the hypotenuse. The lengths of the sides of a right triangle must satisfy the Pythagorean theorem, $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse.

The relationship between the angle measures and the side lengths is known as trigonometry. There are three trigonometric ratios of sides that correspond to an acute angle measure in a right triangle.

Three Trigonometric Ratios

Given the right triangle ABC, where angle A is the right angle shown in the figure, the sides are labeled based upon their relationship to angle θ.



The ratio of sides in the box above can be remembered using the following mnemonic device:

$$\mathrm{S}\frac{\mathrm{O}}{\mathrm{H}} \qquad \mathrm{C}\frac{\mathrm{A}}{\mathrm{H}} \qquad \mathrm{T}\frac{\mathrm{O}}{\mathrm{A}}$$

Trigonometric Ratios of Angle

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AC}{BC}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{BC}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{AC}{AB}$$

Special Right Triangles

Two triangles have known angle measures and corresponding side lengths. These are known as the special right triangles.

30°–60°–90° Triangle

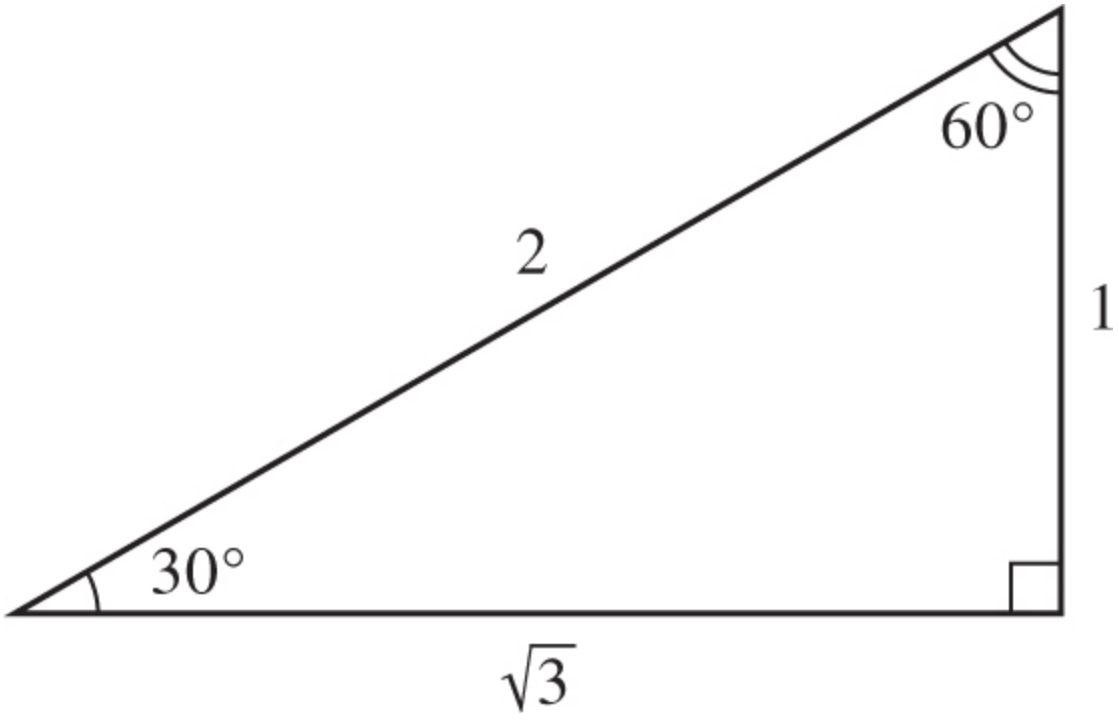
In this triangle, the two acute angles measure 30° and 60° and the third angle is the right angle, 90°. Their corresponding side lengths are in the ratio $1:\sqrt{3}:2$, as shown in the figure below.

As a result, below are six known trigonometric values that correspond to this special right triangle:

$$\sin 30^\circ = \frac{1}{2} \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{1}{2} \qquad \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$$



45°–45°–90° Triangle

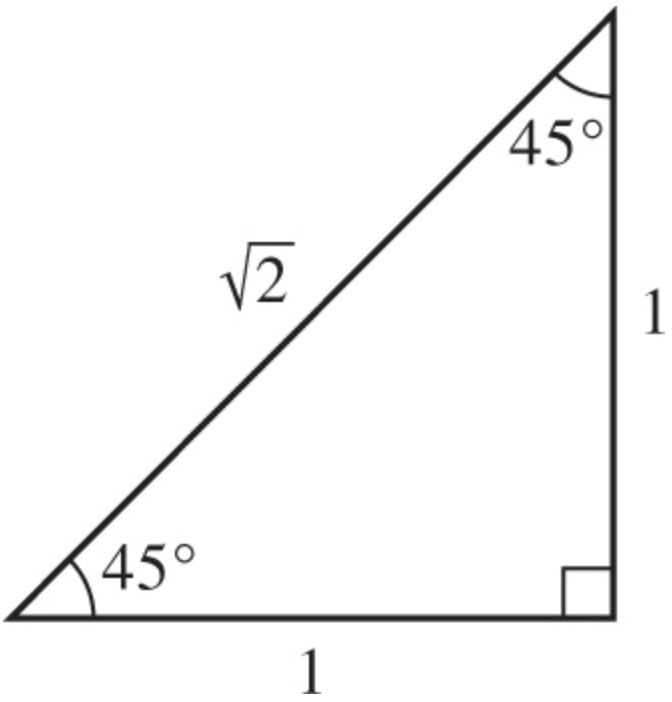
In this triangle, the two acute angles measure 45° and 45° and the third angle is the right angle, 90°. Their corresponding side lengths are in the ratio $1:1:\sqrt{2}$, as shown in the figure.

As a result, below are three known trigonometric values that correspond to this special right triangle:

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{1}{1} = 1$$



The trigonometric values of angles other than 30°, 45°, and 60° can be measured using calculators. In this course, you will learn that the trigonometric values of angles that measure more than 90° can also be evaluated.

Law of Sines and Law of Cosines

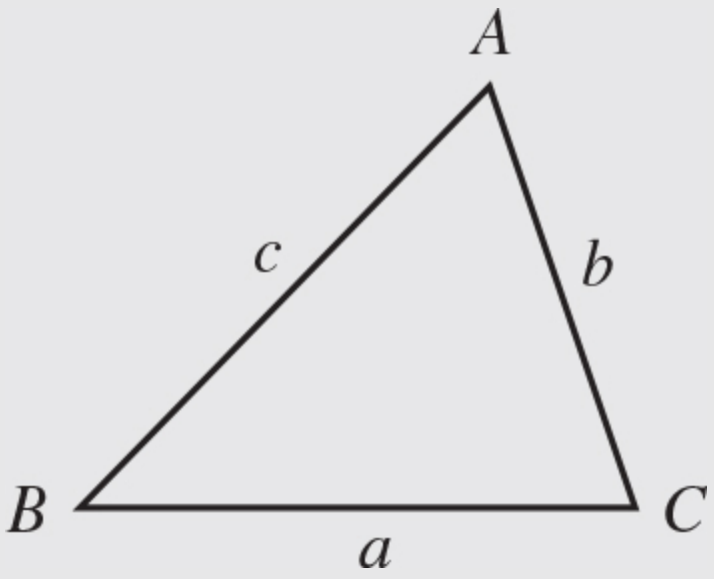
Two laws help determine the lengths and angles of triangles that don't necessarily have to be right triangles. They are known as the law of sines and the law of cosines.

Law of Sines

The law of sines relates the lengths of two sides of a triangle to the sines of the angles opposite those sides.

The Law of Sines

$$\frac{\text{side of } \Delta ABC}{\text{sine of angle opposite side}} = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Example

In ΔABC, $a = 12$, $\sin A = 0.6$, and $\sin B = 0.4$. What is the length of side b?

Solution

Since the sine of two angles and the length of a side are given, solve for b by using the law of sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{12}{0.6} = \frac{b}{0.4}$$

$$0.6b = (0.4)(12)$$

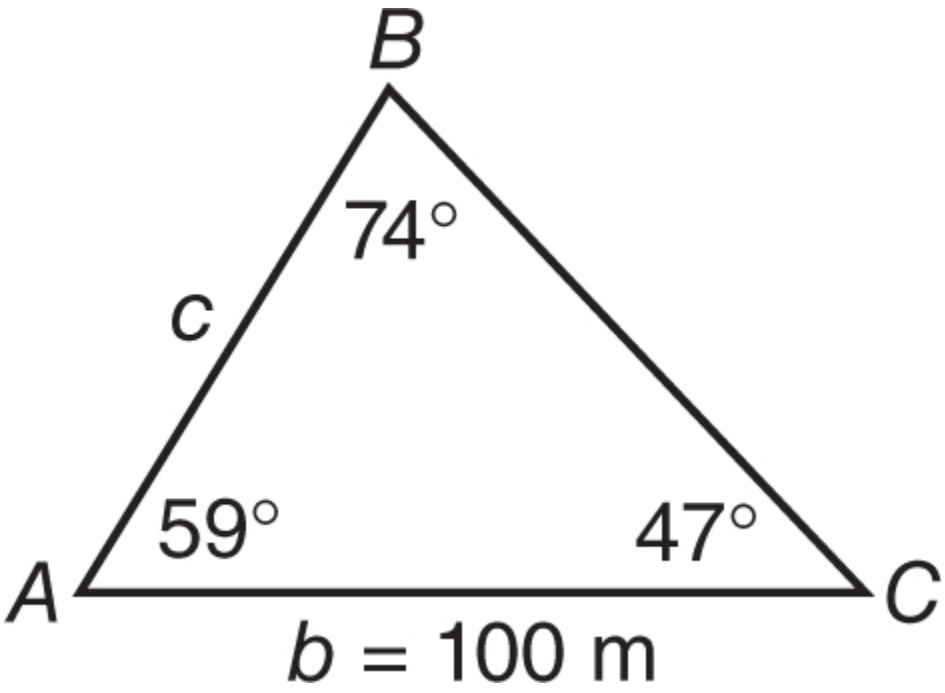
$$b = \frac{4.8}{0.6} = 8$$

Example

In ΔABC, $\angle A = 59°$, $\angle B = 74°$, and $b = 100$ meters. Find c to the nearest tenth of a meter.

☑ Solution

It is helpful to make a model of the given information to determine the necessary equation needed to solve the problem.



First find the measure of $\angle C$, which is the third angle of the triangle:

$$\angle C = 180° - (59° + 74°) = 47°$$

Now find c using the law of sines.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 47°} = \frac{100}{\sin 74°}$$

$$c \sin 74° = (100)(\sin 47°)$$

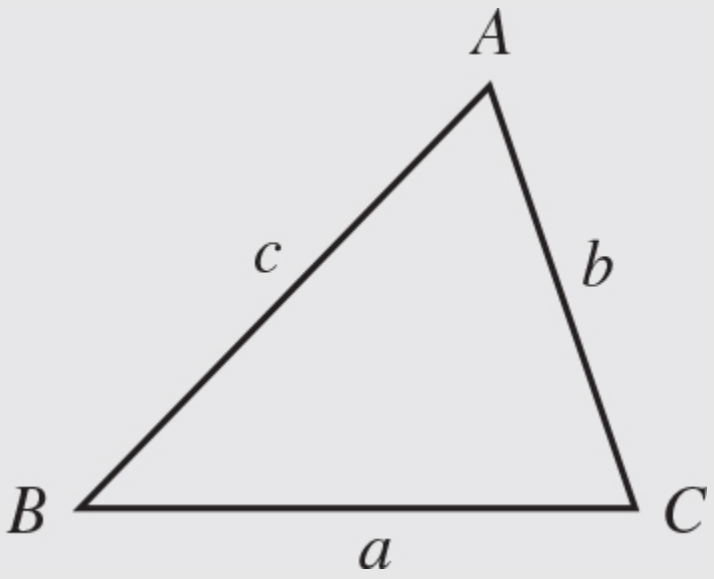
$$c = \frac{100 \sin 47°}{\sin 74°} = 76.1 \text{ m}$$

Law of Cosines

The law of cosines relates the lengths of the three sides of a triangle to the cosine of the angle opposite one of the sides.

The Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

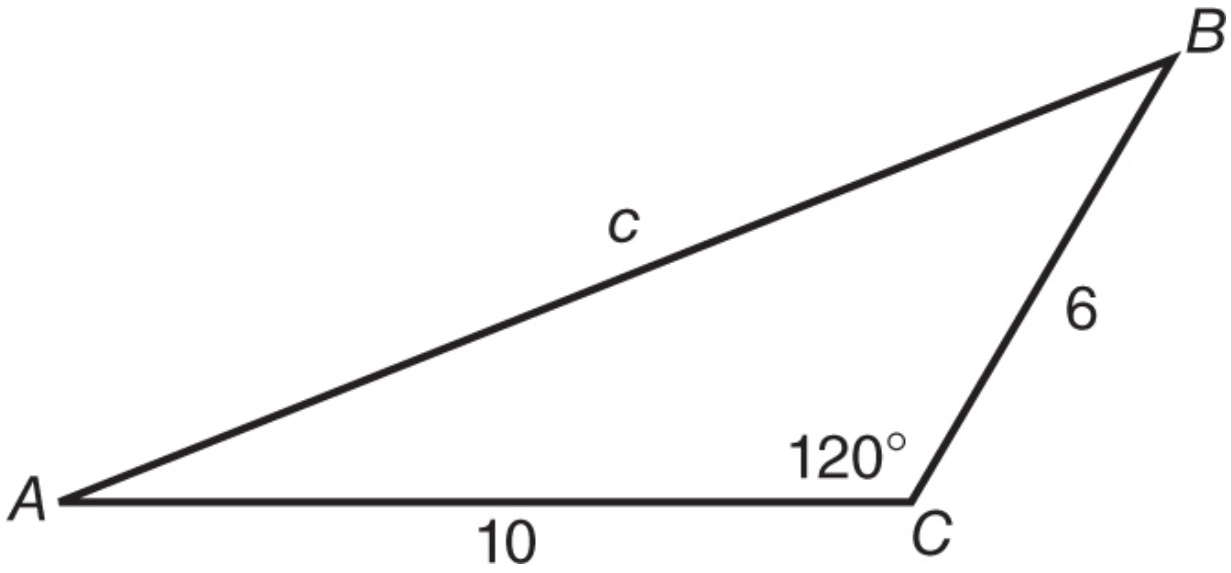


› Example

In ΔABC, $a = 6$, $b = 10$, and $\angle C = 120°$. What is the length of side c?

☑ Solution

It is helpful to make a model of the given information to determine the necessary equation needed to solve the problem.



Since three sides and one angle are marked in the model, use the law of cosines to find the length of side *c*.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = (6)^2 + (10)^2 - 2(6)(10)\cos 120^\circ$$

Use your calculator.

$$c^2 = 196$$

$$c = \pm\sqrt{196} = \pm 14$$

Since *c* represents the length of a side, reject the negative length. Therefore, $c = 14$.

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Unit 1

Polynomial and Rational Functions

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1

Rates of Change

Learning Objectives

In this chapter, you will learn:

- ➜ The difference between a relation and a function
- ➜ How to calculate the rate of change and its meaning
- ➜ How to calculate the rate of change of different functions

1.1 Relations and Functions

Relations

Most mathematical studies are based upon two different quantities and how they relate to one another. This is known as a relation. A relation can be represented by a diagram, an equation, or a list that specifies a specific relationship between groups of elements.

For example, consider relation *r* shown in Figure 1.1 as both a diagram and a list of ordered pairs.

r:

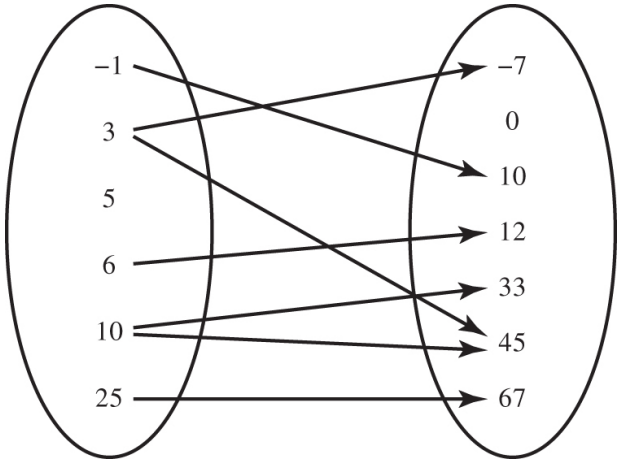


Figure 1.1

The elements in the oval on the left are known as the set of input values, which is also called the domain. The elements in the oval on the right are known as the set of output values, which is also called the range. The arrows lead from the inputs and end at their corresponding outputs.

Following the arrows helps to write the relation as a set of ordered pairs as shown below:

$$r: \{(-1, 10), (3, -7), (3, 45), (6, 12), (10, 33), (10, 45), (25, 67)\}$$

There are two things to notice about relation *r*:

1. Not all of the elements in the domain were used: 5 did not have a corresponding output.
2. Some inputs had multiple outputs: 3 mapped to −7 and 45, and 10 mapped to 33 and 45.

Functions

A relation that maps a set of input values to a set of output values where each input value is mapped to one and only one output value is defined as a function. For relation *r* as a function, 5 would need to be removed from the domain, and both 3 and 10 should be mapped to only one output each. This is shown in Figure 1.2.

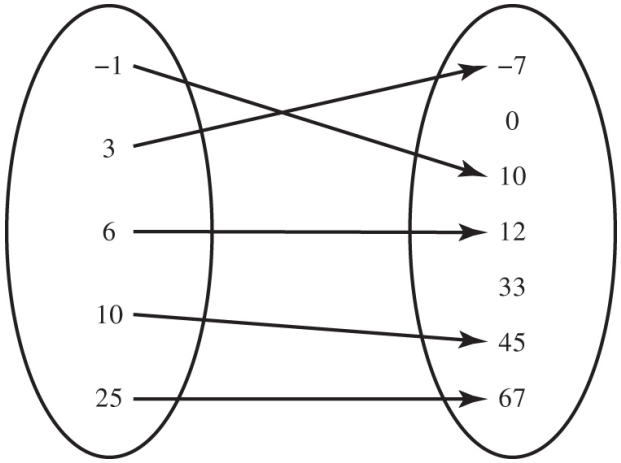


Figure 1.2

An example of a real-life model of a function is the cost of a taxi. Taxis calculate the cost of a ride based on the amount of time a passenger spends in the vehicle. Therefore, the cost of a particular ride is a function of the length of the ride. A passenger cannot have two charges for the same duration. In this example, time is the domain and cost is the range.

Functions are powerful mathematical tools for describing how one variable quantity depends on another. A function can be represented verbally, numerically, algebraically, or graphically. Variables are often used to represent the infinitely many possibilities for the domain and range of a function. The variable representing input values is called the independent variable, and the variable representing output values is called the dependent variable. Since functions, like relations, can be represented by ordered pairs, x is often used to represent the independent variable and y often represents the dependent variable. Then every input and its corresponding output can be represented by the ordered pair (x, y).

Often functions can be written as equations based upon a rule that describes the relationship between the independent and dependent variables. Returning to the cost of the taxi, for a passenger to ride, it costs $2.00 to enter the car and $1.50 for every minute of the ride. A table of values can be created to list the different corresponding inputs and outputs. For example, at 0 minutes, the cost is $2.00 just to enter the taxi. After a minute of driving, $1.50 is added to that cost for a total of $3.50. At 2 minutes,

another $1.50 is added for a total cost of $5.00. This is illustrated in Table 1.1.

Table 1.1

x = time (minutes)	y = cost (dollars)
0	2.00
1	3.50
2	5.00
3	6.50
4	8.00

Since there are infinitely many values that time can take on, listing all the corresponding pairs is tedious. It is more efficient to write the cost of a ride as an equation dependent upon time. Since the independent variable is time, let x represent the time in minutes and y represent the cost in dollars. The cost equation could be written as $y = 2.00 + 1.50x$.

There is another way to notate functions as equations. If f represents a function, then $f(x)$, read as "f of x," represents the value of the function when x takes on a specific value. In other words, $y = f(x)$. The benefit of this notation is that it allows you to see the two variables and clearly identify the independent and dependent variables. It is also a nice way to identify specific values. For example, to find the cost of the taxi ride after 5 minutes, we would input 5 for x to find the cost. This substitution would look like $f(5) = 2.00 + 1.50(5) = 9.50 = y$. After 5 minutes, the cost of the taxi would be $9.50.

Different letters can be used to fit what the variables represent. For example, t can stand for time and c for cost. Our taxi equation can be rewritten as $c(t) = 2.00 + 1.50t$.

› Example

If $f(x) = 10x - 2$, find the value of $f(3)$.

✓ Solution

To find the value of $f(3)$, the input is $x = 3$. This value is substituted for x in the equation.

$$f(3) = 10(3) - 2 = 30 - 2 = 28$$

When $x = 3$, $f(3) = 28 = y$. The ordered pair that represents this solution is (3, 28).

› Example

Given $g = \{(-1, 1), (0, 3), (4, 7), (5, 1), (8, -10)\}$, find $g(0)$.

✓ Solution

To find the value of $g(0)$, the input is $x = 0$. Find the ordered pair whose x-value is 0. The y-value will represent the function value. The ordered pair is (0, 3), so $g(0) = 3$.

› Example

If $v(t) = t^2 + 5$, what is $v(c + 2)$?

✓ Solution

To evaluate $v(c + 2)$, the input is $t = c + 2$. This input is substituted for t in the equation.

$$v(c+2) = (c+2)^2 + 5 = (c+2)(c+2) + 5 = c^2 + 2c + 2c + 4 + 5 = c^2 + 4c + 9$$

Another way functions can be represented is by their graphs. To graph a function, plot its coordinate pairs as points on an xy-plane. The xy-plane is a coordinate grid with a horizontal line that represents the x-axis and a

vertical line that represents the *y*-axis. Coordinate pairs are plotted by moving left or right on the *x*-axis and then moving down or up on the *y*-axis. If the function is continuous over its domain, the points of the pairs can be connected to reveal the graph. The graph provides a nice visual and can show different behaviors of the graph that can lead to important conclusions about the function.

Returning to the taxi cost function, $f(x) = 2.00 + 1.50x$, graph the function on the coordinate plane. Begin by plotting the values from Table 1.2 on the coordinate plane, as shown in Figure 1.3.

Table 1.2

x = time (minutes)	y = cost (dollars)
0	2.00
1	3.50
2	5.00
3	6.50
4	8.00

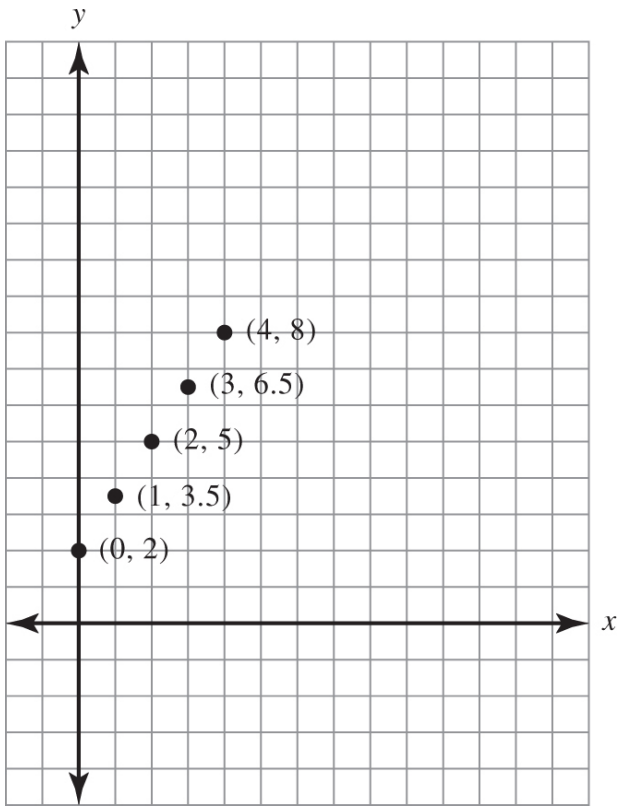


Figure 1.3

Since the domain is time, assume the inputs are all real numbers greater than zero. The points can be connected to create a continuous graph, as shown in Figure 1.4.

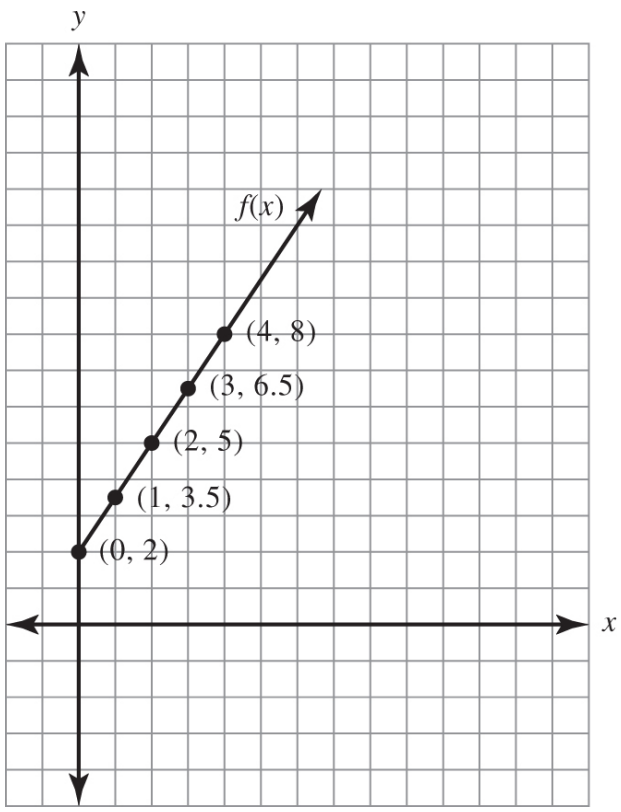


Figure 1.4

This is a linear function because its graph is a line. Note that the axes can change names, too, depending on the independent and dependent variables being used. If we graphed $c(t) = 2.00 + 1.50t$, the x-axis could be renamed as t and the y-axis could be renamed as either c or $c(t)$.

Various types of functions will be explored in later chapters of this book. Many have general concepts that can be applied, including domain, range, x- and y-intercepts, and graph behaviors.

Domain and Range

The domain of a function, $f(x)$, is the largest possible set of numbers x for which $f(x)$ is a real number. To find the domain graphically, look along the x-axis from left to right. The x-value of the point farthest to the left represents the lower bound of the domain, and the x-value of the point farthest to the right represents the upper bound of the domain.

The range of a function, $f(x)$, is the set of all values that y can have as x takes on each of its possible values.

Figure 1.5 is the graph of function $g(x)$.

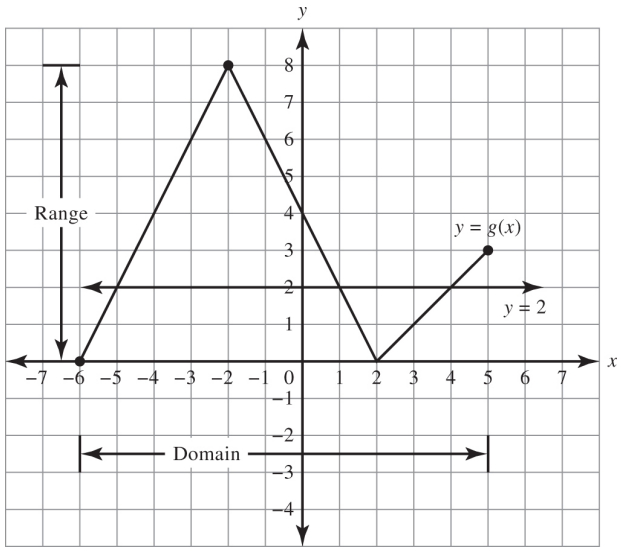


Figure 1.5

The point farthest to the left is (−6, 0), and the point farthest to the right is (5, 3). Therefore, the domain of $g(x)$ is $-6 \le x \le 5$, which can be written as the interval [−6, 5].

The lowest points are (−6, 0) and (2, 0). The highest point is (−2, 8). Therefore, the range of $g(x)$ is $0 \le y \le 8$, which can be written as the interval [0, 8].

Figure 1.5 has closed circles on the endpoints, but not all graphs have closed endpoints, as seen in the next figure.

Figure 1.6 is the graph of function $h(x)$.

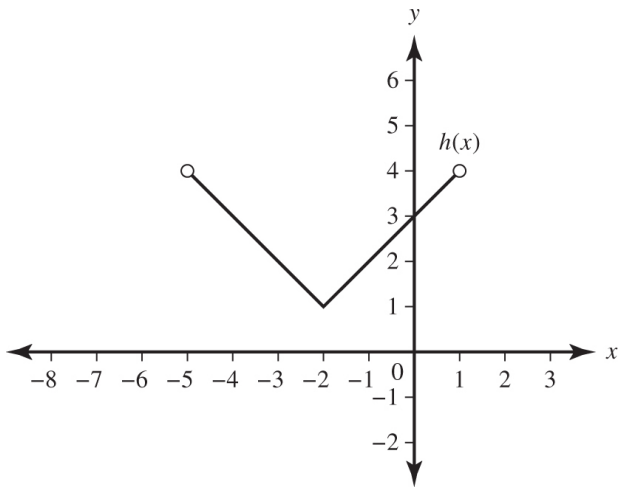


Figure 1.6

The point farthest to the left is (−5, 4), and the point farthest to the right is (1, 4). Since these points have open circles, it means the graph of the function goes up to those points but does not include those values. Therefore, the domain of $h(x)$ is $-5 < x < 1$, which can be written as the interval (−5, 1).

The lowest point is (−2, 1) and is graphed with a closed circle. The highest points are (−5, 4) and (1, 4), which are graphed with open circles. Therefore, the range of $h(x)$ is $1 \leq y < 4$, which can be written as the interval [1, 4).

Figure 1.7 is the graph of $k(x)$.

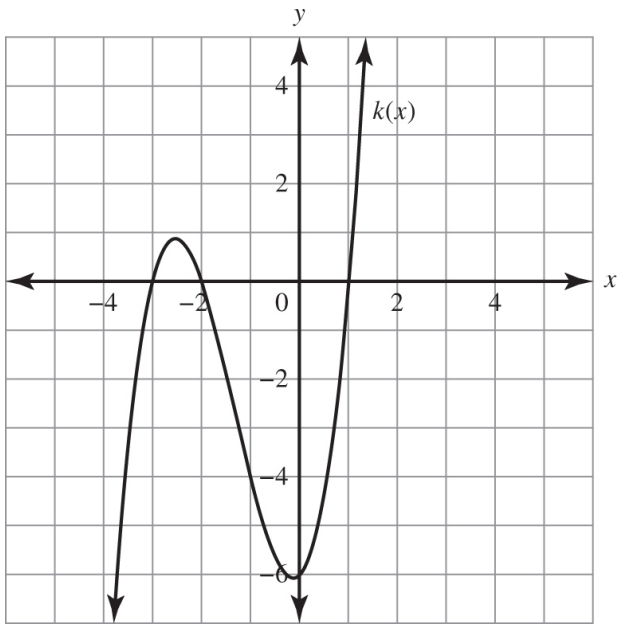


Figure 1.7

There are arrowheads on both endpoints. This means the graph continues in those directions forever. Since there is no left or right endpoint, the domain is all real numbers, which can be written in interval notation as $(-\infty, \infty)$. Since there is no lowest or highest endpoint, the range is all real numbers, which is written in interval notation as $(-\infty, \infty)$.

Example

State the domain and range of each of the functions in Figures 1.8 and 1.9.

1.

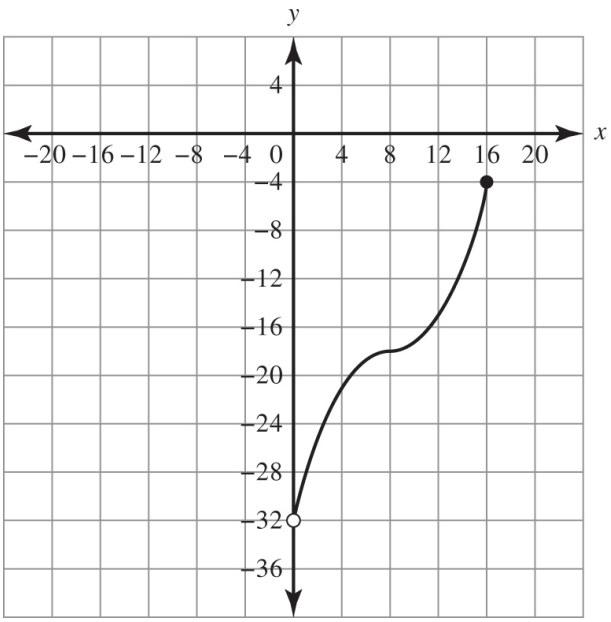


Figure 1.8

2.

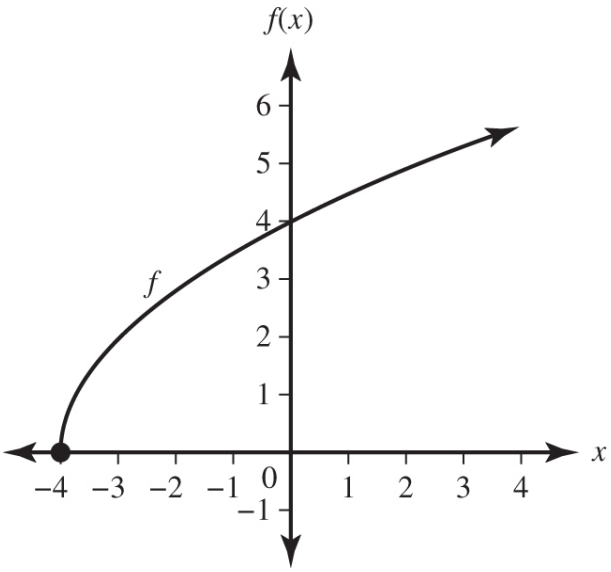


Figure 1.9

☑ Solution

1. The point farthest to the left is (0, −32) and has an open circle. The point farthest to the right is (16, −4) and has a closed circle. Therefore,

the domain is $0 < x \leq 16$, which can be written as the interval $(0, 16]$.

The lowest point is $(0, -32)$ and has an open circle. The highest point is $(16, -4)$ and has a closed circle. Therefore, the range is $-32 < y \leq -4$, which can be written as the interval $(-32, -4]$.

2. The point farthest to the left is $(-4, 0)$ and has a closed circle. There is no point farthest to the right since there is an arrowhead on the right side of the graph. Therefore, the domain is $-4 \leq x$ or $x \geq -4$, which can be written as the interval $[-4, \infty)$.

   The lowest point is $(-4, 0)$ and has a closed circle. There is no highest point since there is an arrowhead on the right side of the graph. Therefore, the range is $0 \leq y$ or $y \geq 0$, which can be written as the interval $[0, \infty)$.

Algebraically, the domain can be calculated by considering the input values that would make the function not exist or be undefined.

Rules to Find the Domain

There are two key rules to follow when algebraically finding the domain of a function.

1. Do not divide by zero. Exclude from the domain of a function any value of x that results in a 0 in the denominator. For example, if $f(x) = \frac{x+4}{x-3}$, then $x \neq 3$ since $f(3) = \frac{3+4}{3-3} = \frac{7}{0}$, which is undefined. Therefore, the domain of the function is the set of all real numbers except 3, which can be represented as $(-\infty, 3) \cup (3, \infty)$.
2. Do not take the square root (or any even root) of a negative number. Since the square root of a negative number is not a real number, the quantity underneath a square root must always be greater than or equal to 0. If $f(x) = \sqrt{x+5}$, then $x + 5 \geq 0$ and $x \geq -5$. The domain of the function is the set of all real

numbers greater than or equal to −5, which can be represented as [−5, ∞).

Intercepts

The x-intercept of the graph of a function is found where the graph intersects the x-axis. At this point, the output value is zero and the corresponding input values are said to be the zeros of the function. When graphically finding the zeros of a function, look to see where the graph intersects the x-axis.

The y-intercept of the graph of a function is found where the graph intersects the y-axis. At this point, the input value is zero.

Example

Find the zeros of the function $f(x)$ shown in Figure 1.10. Also state the coordinates of the y-intercept.

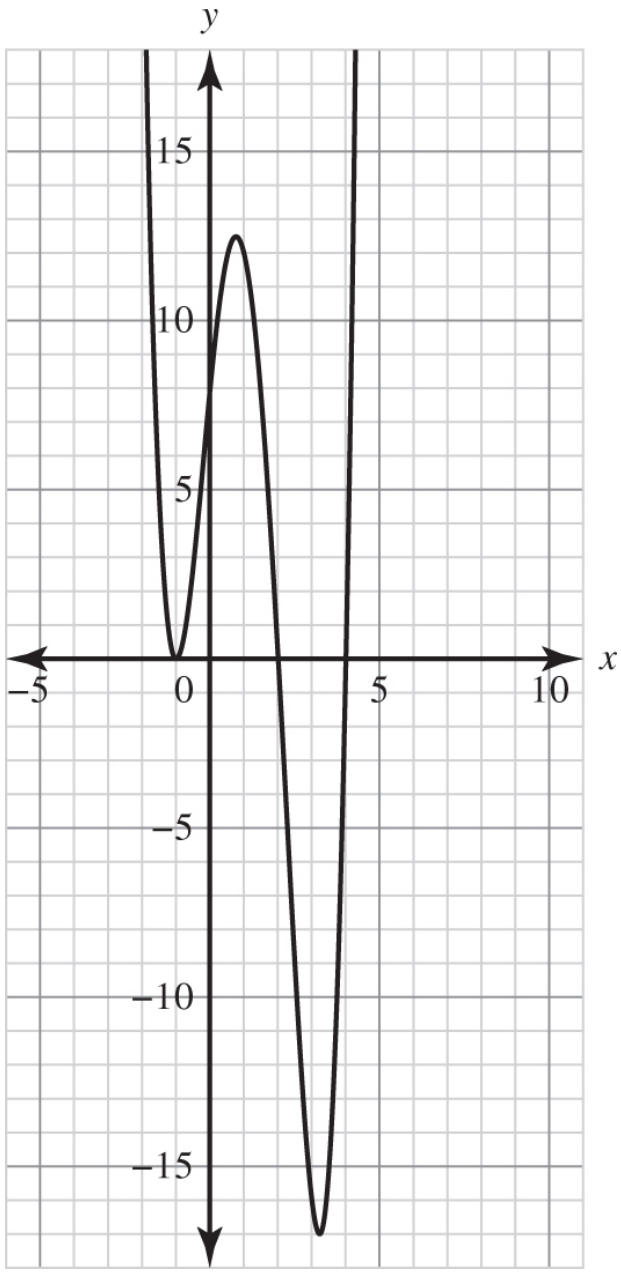


Figure 1.10

☑ Solution

To find the zeros, locate the x-intercepts. The x-intercepts are the points $(-1, 0)$, $(2, 0)$, and $(4, 0)$. Therefore, the zeros are $x = -1$, 2, and 4.

The y-intercept is the point $(0, 8)$.

To find the zeros of a function algebraically, set the equation equal to zero and solve for x. The x-values will represent the zeros of the function.

To find the y-intercept of a function algebraically, substitute 0 for the input value and solve for y. The y-value will represent the y-coordinate of the y-intercept.

› Example

Find the zeros of each of the following functions.

1. $f(x) = 2x - 48$
2. $g(x) = (x - 3)(x + 5)(x + 13)$

✓ Solution

1. To find the zeros of the function, first set the equation equal to 0. Solve for x by adding 48 to both sides of the equation and then dividing both sides by 2.

$$\begin{aligned} 2x - 48 &= 0 \\ 2x &= 48 \\ x &= 24 \end{aligned}$$

2. First set the equation equal to 0. To solve for x, set each of the factors equal to 0 since the only way a product can equal 0 is if at least one of the terms in the parentheses is 0.

$$(x - 3)(x + 5)(x + 13) = 0$$

$$\begin{aligned} (x - 3) &= 0 \\ x &= 3 \end{aligned} \qquad \begin{aligned} (x + 5) &= 0 \\ x &= -5 \end{aligned} \qquad \begin{aligned} (x + 13) &= 0 \\ x &= -13 \end{aligned}$$

› Example

Find the y-intercept of each of the following functions.

1. $f(x) = 2x - 48$
2. $g(x) = (x - 3)(x + 5)(x + 13)$

✓ Solution

1. To find the y-intercept of a function, substitute 0 for the input value and solve for y.

$$f(0) = 2(0) - 48 = 0 - 48 = -48 = y$$

The y-intercept has the coordinates $(0, -48)$.

2. Substitute 0 for the input value, and solve for y.

$$g(0) = (0 - 3)(0 + 5)(0 + 13) = (-3)(5)(13) = -195 = y$$

The y-intercept has the coordinates $(0, -195)$.

Notice in the previous two examples that a function can have multiple zeros or x-intercepts but only one y-intercept. This is true for every function since a function can have one and only one output for any input. Since the input for a y-intercept is always 0, that 0 can correspond to only one y-value in order to remain a function.

Remember that a function can have many x-intercepts or zeros but can have only one y-intercept.

This rule can easily be visualized using a graph and is called the vertical line test. The vertical line test states that an equation represents a function if no vertical line intersects the graph of the equation in more than one point.

The graph in Figure 1.11 represents a function since any vertical line that can be drawn intersects the graph in at most one point. The graph in Figure 1.12 is not a function since it fails the vertical line test.

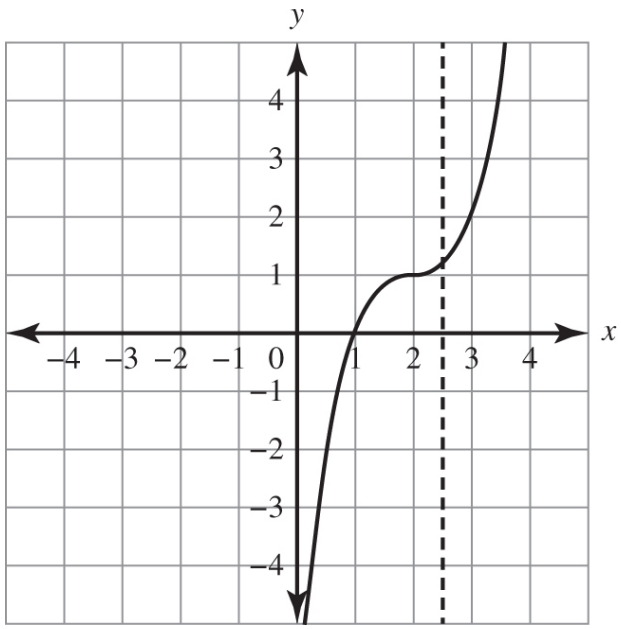


Figure 1.11

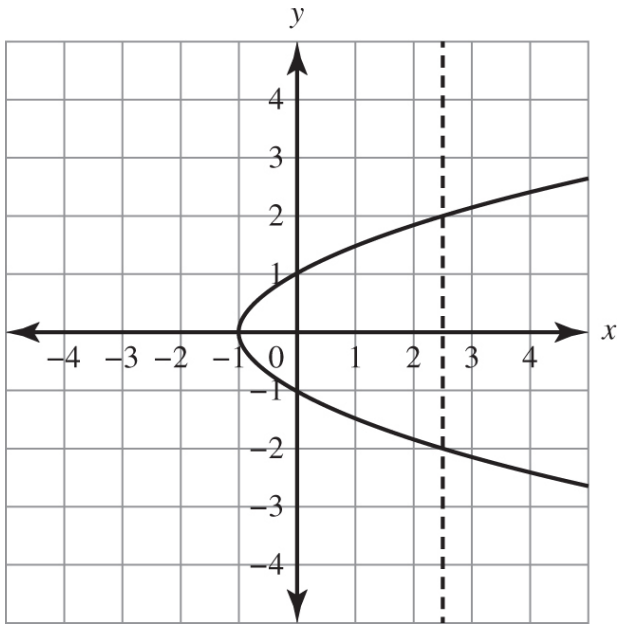


Figure 1.12

Additional Behaviors of Functions

The last general behaviors that will be discussed about functions can be easily viewed from a function's graph and help describe its shape. These

behaviors can also be tested algebraically and are a major topic of conversation in calculus.

Increasing/Decreasing

A function is increasing over an interval of its domain if, as the input values increase, the output values always increase. In other words, for all a and b in the interval of the domain, if $a < b$, then $f(a) < f(b)$.

A perfect example of an increasing function is the taxi cost equation, $f(x) = 2.00 + 1.50x$. As time increases, the cost of the taxi continues to increase. As shown in Figure 1.4, as you follow the path of the graph from left to right, the graph continues to travel up.

Alternatively, a function is decreasing over an interval of its domain if, as the input values increase, the output values always decrease. In other words, for all a and b in the interval, if $a < b$, then $f(a) > f(b)$.

A real-world example that models a decreasing function is the value of a new car. After you buy a car, as time increases, the value of the car continues to decrease as the car is said to depreciate.

Figure 1.13 shows the graph of an increasing function, and Figure 1.14 shows the graph of a decreasing function.

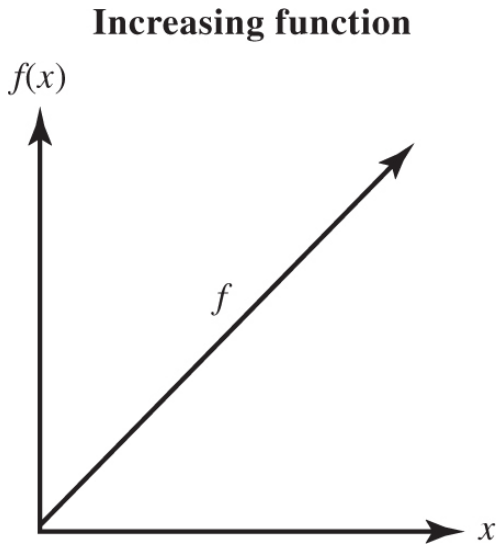


Figure 1.13

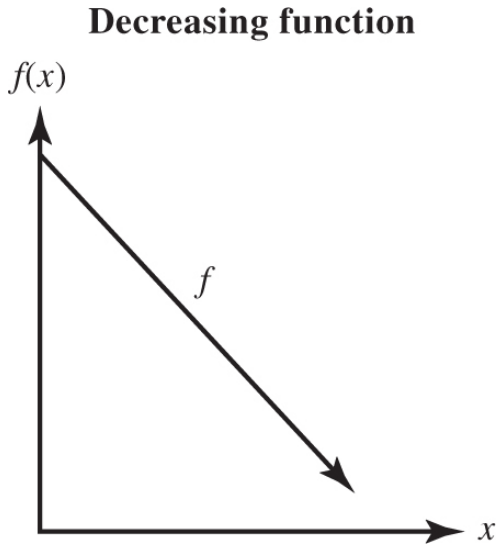


Figure 1.14

Note that a graph of a function can have both increasing and decreasing intervals over the domain.

Example

Using the graph of the function shown in Figure 1.15, identify the intervals where the function is increasing and the intervals where it is decreasing.

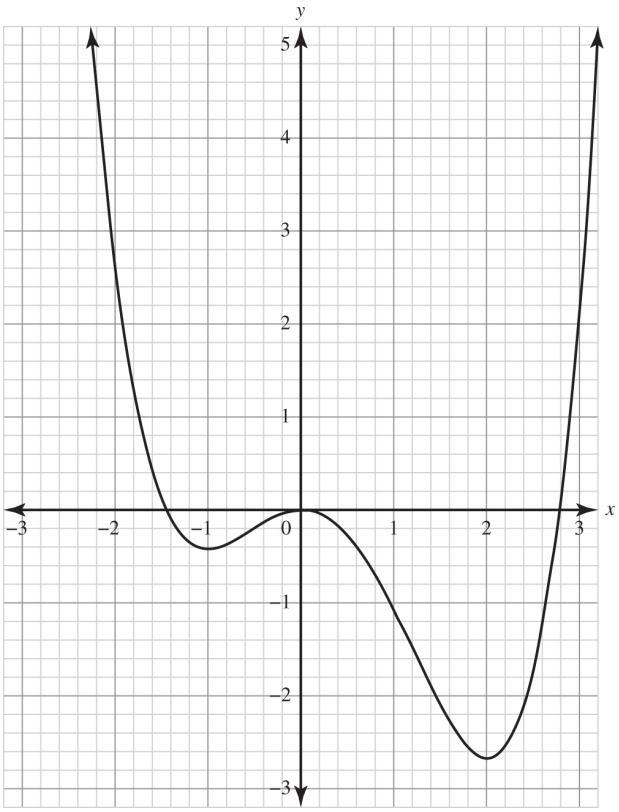


Figure 1.15

☑ Solution

Begin at the left endpoint of the graph, and follow its path to the right endpoint of the graph. Note that the function changes behavior more than once. When stating the intervals, use the *x*-values since they represent the independent variable.

The function is decreasing over the intervals $(-\infty, -1] \cup (0, 2)$.

The function is increasing over the intervals $(-1, 0) \cup (2, \infty)$.

Concavity

Unlike linear functions, the graphs of other functions that will be discussed are not only continuous but are also smooth and appear to bend, sometimes upward or downward. This is known as the concavity of a graph, as shown in Figure 1.16.

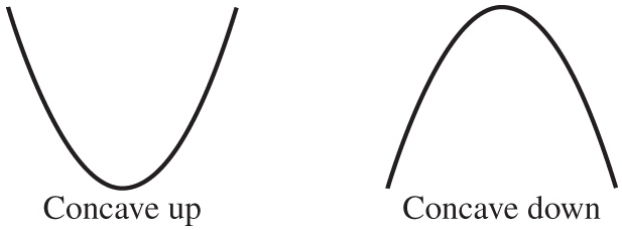


Figure 1.16

A more formal definition of concavity now leads to a discussion of rates of change.

1.2 Rates of Change

A rate is a comparison between two different quantities.

To find the average rate of change, either two points or two sets of inputs with their corresponding outputs are needed. Then the differences can be calculated.

Finding the average rate of change between an initial point and a final point is equivalent to finding the slope of the segment (or secant) between those points.

Average Rate of Change

The average rate of change is the change in the value of one quantity divided by the change in the value of another quantity. Common examples of average rate of change include finding the average $\text{speed} = \dfrac{\text{change in distance}}{\text{change in time}}$ and the slope of a line $\text{segment} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Example

A car travels between two cities according to the distance-time graph shown in Figure 1.17.

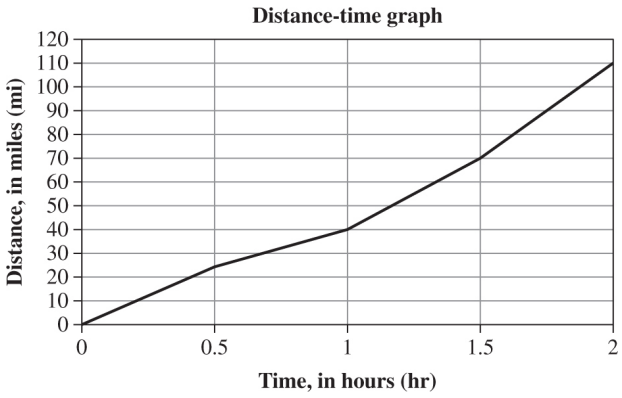


Figure 1.17

(a) What is the car's average speed for the first hour?

(b) What is the car's average speed for the entire trip?

(c) How does the average speed in part (a) compare with the average speed in part (b)?

✓ Solution

(a) Initially, the car has traveled 0 miles in 0 hours. After 1 hour, the car has traveled 40 miles. The average speed for the first hour is

$$\frac{40-0}{1-0} = 40 \text{ mph}$$

(b) Initially, the car has traveled 0 miles in 0 hours. By the end of the trip, the car has traveled 110 miles in 2 hours. The average speed for the entire trip is

$$\frac{110-0}{2-0} = 55 \text{ mph}$$

(c) Figure 1.18 shows the two secants that represent the solutions to parts (a) and (b). The slope of the secant for part (b) is steeper than the slope of the secant for part (a). In other words, over the course of the entire trip, more distance was gained after the initial hour.

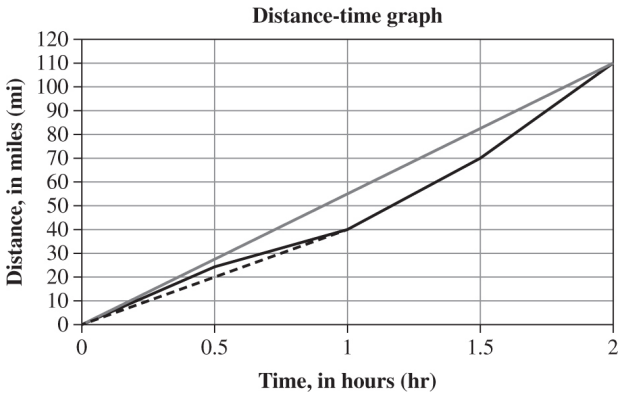


Figure 1.18

The average speed for the entire trip does not give all the details for the speeds along the way. The average speed is just that, an average or an overall idea of the rate of change between a starting point and an ending point.

Finding the speed at a specific instant in time is something entirely different. Graphically at one point, such as $t = 1$ hour, the secant line would change to a tangent line, as shown in Figure 1.19.

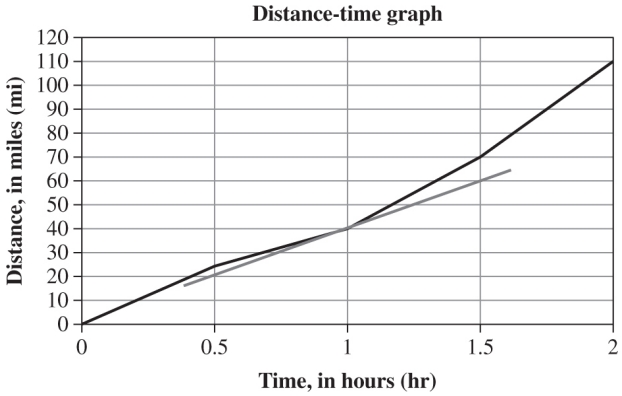


Figure 1.19

Finding the rate of change at this value is an issue since the previous slope formula needs two points and only one point is known for the tangent line.

The rate of change at a point can be approximated by the average rate of change of the function over small intervals containing the point, if such values exist.

Example

Given the graph of function $f(x) = x^2$ shown in Figure 1.20, find the average rate of change at point $P(4, 16)$.

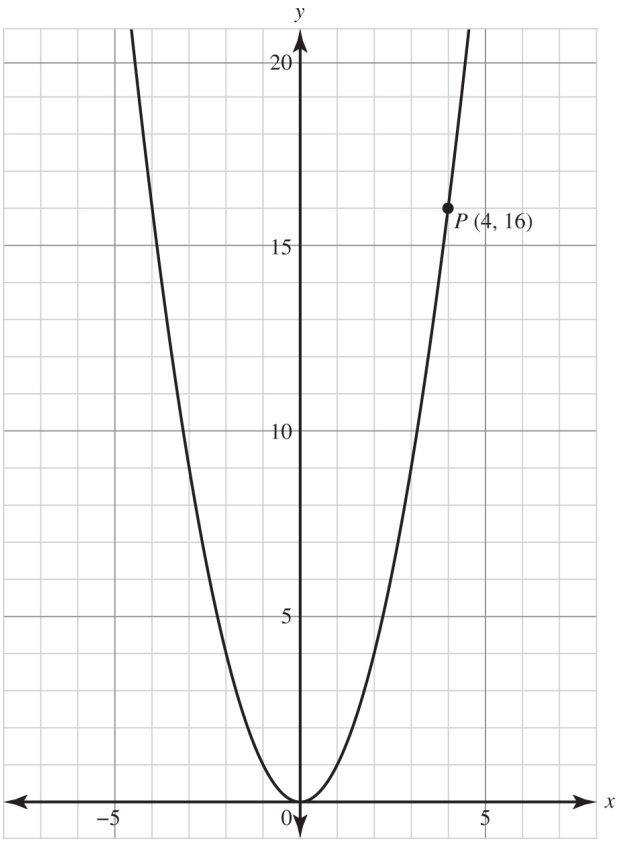


Figure 1.20

Solution

It is impossible to find the average rate of change at one point since two points are needed. To approximate, we will choose points to serve as endpoints with point P to create secant lines. As the endpoints get closer and closer to point P, the slopes of the secant lines should approach one value. This value will be the approximation of the average rate of change at point P. As seen in Figure 1.21, as the endpoints get closer to point P, the secant lines begin to look the same.

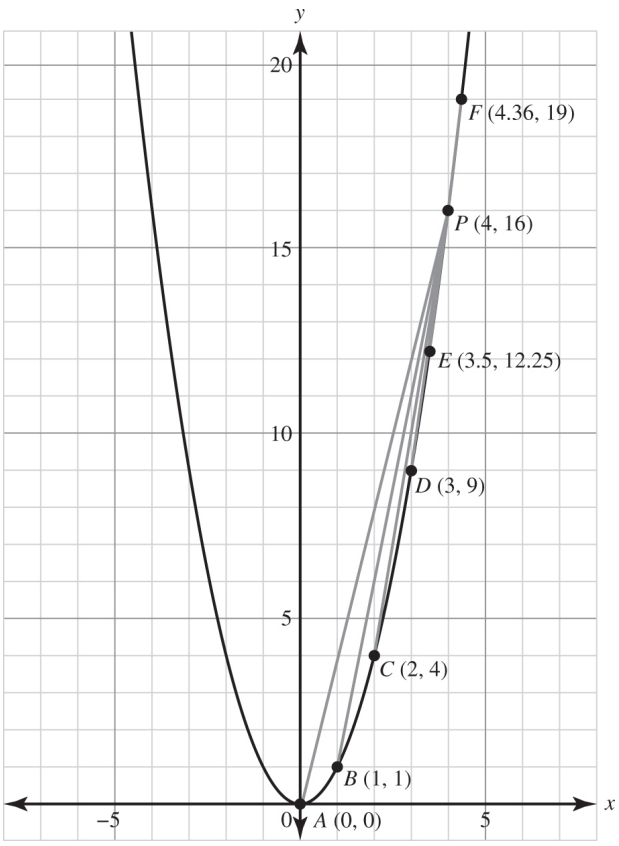


Figure 1.21

Table 1.3 will help to organize the different average rates of change.

Table 1.3

Point	*P*(4, 16)	Average Rate of Change
A(0, 0)	*P*(4, 16)	$\frac{16-0}{4-0}=\frac{16}{4}=4$
B(1, 1)	*P*(4, 16)	$\frac{16-1}{4-1}=\frac{15}{3}=5$
C(2, 4)	*P*(4, 16)	$\frac{16-4}{4-2}=\frac{12}{2}=6$
D(3, 9)	*P*(4, 16)	$\frac{16-9}{4-3}=\frac{7}{1}=7$
E(3.5, 12.25)	*P*(4, 16)	$\frac{16-12.25}{4-3.5}=\frac{3.75}{0.5}=7.5$

Point	*P*(4, 16)	Average Rate of Change
F (4.36, 19)	*P*(4, 16)	$\frac{16-19}{4-4.36}=\frac{-3}{-0.36}=8.\overline{333}$

For *x*-values smaller than 4, the slopes approach 7.5. For *x*-values greater than 4, the values approach $8.\overline{333}$.

Since the equation of the function is given, *x*-values that are even closer to 4 from both the left and right directions can be substituted into the function as shown in Table 1.4.

Table 1.4

Point	*P*(4, 16)	Average Rate of Change
G(3.9, 15.21)	*P*(4, 16)	$\frac{16-15.21}{4-3.9}=\frac{0.79}{0.1}=7.9$
H(3.95, 15.6025)	*P*(4, 16)	$\frac{16-15.6025}{4-3.95}=\frac{0.3975}{0.05}=7.95$
I(3.99, 15.9201)	*P*(4, 16)	$\frac{16-15.9201}{4-3.99}=\frac{0.0799}{0.01}=7.99$
J(4.001, 16.008001)	*P*(4, 16)	$\frac{16-16.008001}{4-4.001}=\frac{-0.008001}{-0.001}=8.001$
K(4.01, 16.0801)	*P*(4, 16)	$\frac{16-16.0801}{4-4.01}=\frac{-0.0801}{-0.01}=8.01$
L(4.1, 16.81)	*P*(4, 16)	$\frac{16-16.81}{4-4.1}=\frac{-0.81}{-0.1}=8.1$

The rates of change at two points can be approximated by calculating average rate of change approximations over sufficiently small intervals containing each point, if such values exist.

Coming in from the left and getting closer to point $P(4, 16)$, points G, H, and I have slopes that approach 7.99. Coming in from the right and getting closer to point $P(4, 16)$, points J, K, and L have slopes that approach 8.001. Overall, it seems that the slopes are approaching the value of 8. Therefore, the approximate average rate of change at point P is 8.

Example

A small local restaurant sponsored a contest to name their new special dish. Patrons submitted their entries between noon ($t = 0$) and 10 P.M. ($t = 10$). The number of entries t hours after noon is modeled by the function N for $0 \le t \le 10$. Values of $N(t)$, at various times t, are shown in Table 1.5. Use the data in the table to approximate the rate at which entries were being deposited at time $t = 8$.

Table 1.5

t (hours)	0	3	6	7	9	10
$N(t)$ (number of entries)	0	25	43	50	79	125

Solution

Since $t = 8$ is between $t = 7$ and $t = 9$, which are included in the table, find the average rate of change between these two points to use as an approximate rate of change for the entries being deposited at time $t = 8$.

Calculate the approximate average rate of change at $t = 8$:

$$\frac{N(9) - N(7)}{9 - 7} = \frac{79 - 50}{2} = \frac{29}{2} = 14.5$$

As demonstrated in the previous examples, a rate of change quantifies how two quantities vary together.

A positive rate of change indicates that as one quantity increases or decreases, the other quantity does the same. Algebraically, when calculating the average rate of change, the value simplifies to a positive number. In the previous example, as time increased from $t = 0$ to $t = 10$, the number of entries continuously increased from $N = 0$ to $N = 125$. In other words, the rate at which entries were being deposited increased as the day went by. A positive rate of change can be seen on the graph shown in Figure 1.22.

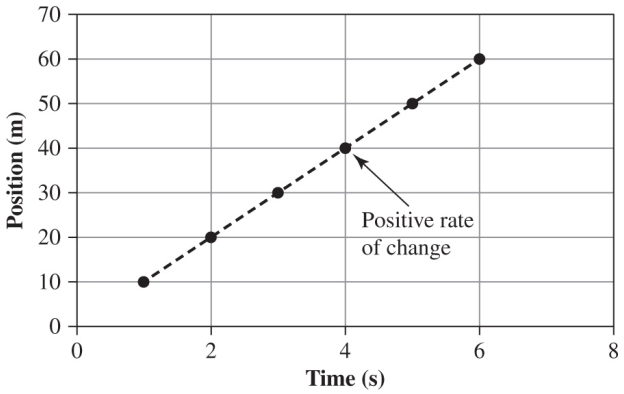


Figure 1.22

A negative rate of change indicates that as one quantity increases, the other decreases. For example, a freshman in college begins the semester with $1,000 in their savings account. Each week they take out $75 to pay for their expenses. After one week, the student has $925. The average rate of change is $\frac{925 - 1000}{1 - 0} = \frac{-75}{1} = -75$. Graphically, a negative rate of change can be seen in Figure 1.23.

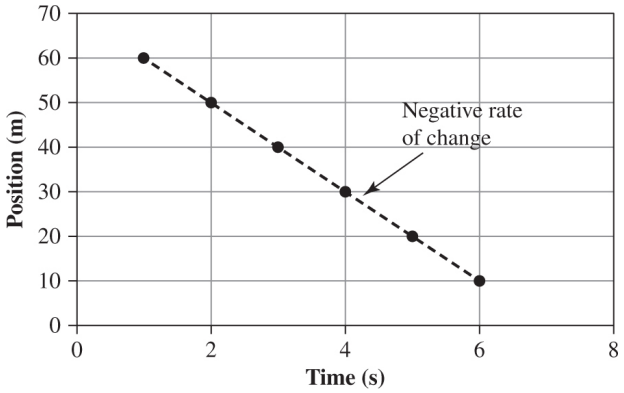


Figure 1.23

Unlike Figures 1.22 and 1.23, not all graphs are linear. Each type of function that will be discussed in later chapters has its own unique graphical behaviors. One such behavior is known as concavity. The concavity of a function describes how the graph of the function curves and bends as shown in Figure 1.24.

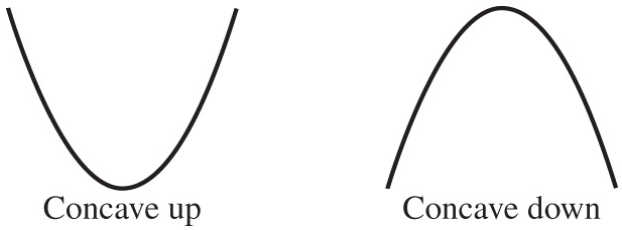


Figure 1.24

More formally, the graph of a function is concave up on intervals in which the rate of change is increasing. To illustrate this, consider the function $f(x) = (x - 4)^2 + 2$, as shown in Figure 1.25 and Table 1.6.

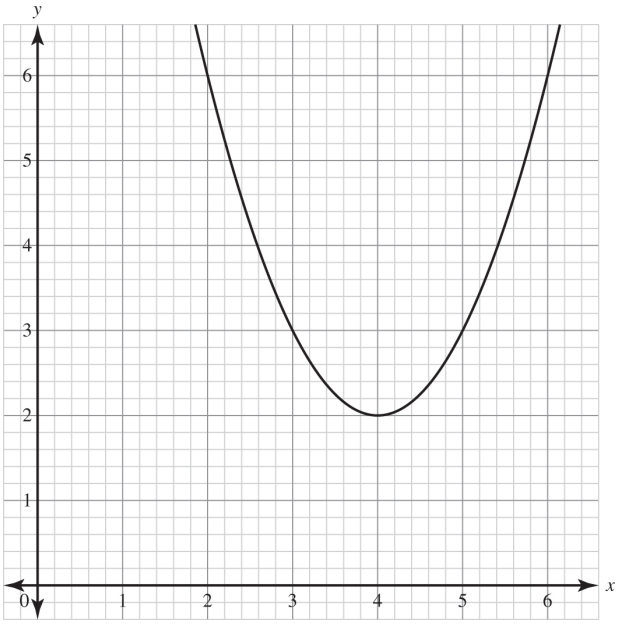


Figure 1.25

Table 1.6

x	$f(x)$
2	6
3	3
4	2
5	3
6	6

Table 1.7 calculates the average rates of change between the pairs of points.

Table 1.7

Point 1	Point 2	Average Rate of Change
(2, 6)	(3, 3)	−3
(3, 3)	(4, 2)	−1
(4, 2)	(5, 3)	1
(5, 3)	(6, 6)	3

As x is increasing, the average rate of change is also increasing. This illustrates why the graph of the function is concave up in Figure 1.25.

Alternatively, the graph of a function is concave down on intervals in which the rate of change is decreasing. To illustrate this, consider the function $f(x) = -(x - 4)^2 + 2$, as shown in Figure 1.26 and Table 1.8.

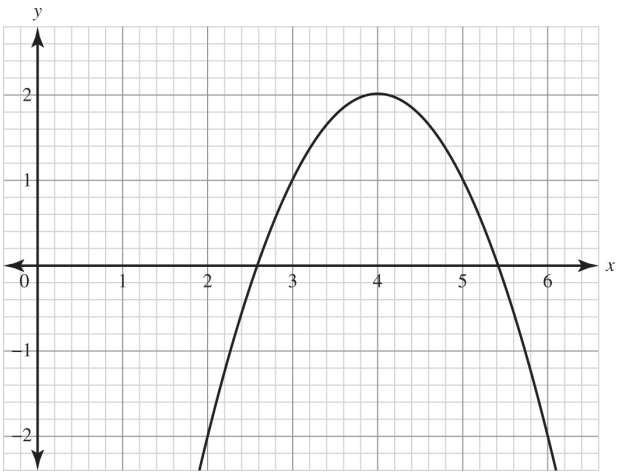


Figure 1.26

Table 1.8

x	$f(x)$
2	−2
3	1
4	2
5	1
6	−2

Table 1.9 calculates the average rates of change between the pairs of points.

Table 1.9

Point 1	Point 2	Average Rate of Change
(2, −2)	(3, 1)	3
(3, 1)	(4, 2)	1

Point 1	Point 2	Average Rate of Change
(4, 2)	(5, 1)	−1
(5, 1)	(6, −2)	−3

As x is increasing, the average rate of change is decreasing. This illustrates why the graph of the function is concave down in Figure 1.26.

1.3 Rates of Change in Linear Functions

Many different types of functions will be discussed and analyzed in this course. Calculating the average rate of change can be used to help identify some of them. Consider Table 1.10, which shows the ordered pairs of the linear function $y = f(x) = -4x + 7$.

Table 1.10

x	$y = f(x)$	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
−2	15	$= \frac{11-15}{-1-(-2)} = \frac{-4}{1} = -4$
−1	11	$= \frac{7-11}{0-(-1)} = \frac{-4}{1} = -4$
0	7	$= \frac{3-7}{1-0} = \frac{-4}{1} = -4$
1	3	$= \frac{-1-3}{2-1} = \frac{-4}{1} = -4$
2	−1	$= \frac{-5-(-1)}{3-2} = \frac{-4}{1} = -4$
3	−5	

Rate of Change

In general, all linear functions have a constant rate of change, $m = \frac{\Delta y}{\Delta x}$, and can be written in the slope-intercept form $y = f(x) = mx + b$, where b is the y-coordinate of the y-intercept. Only linear functions have a constant average rate of change. The

average rate of change can be used to determine whether a particular function is linear or not. Functions that are not linear are said to be nonlinear.

As the value of x increases by 1, the value of y decreases by 4. In other words, for every consecutive integer ordered pair, the average rate of change is −4. This would also be true over any interval length. Consider the ordered pairs (−2, 15) and (3, −5). The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{-5-15}{3-(-2)} = \frac{-20}{5} = -4$$

It is not a coincidence that the average rate of change of the linear function $f(x) = -4x + 7$ is the slope of the linear function.

Example

A strain of bacteria is placed into a Petri dish and allowed to grow. The data collected are shown in Table 1.11. The population is measured in kilograms and the time in hours. The data in Table 1.12 represent the maximum number of hours an individual should work during the week. For both tables, use the average rate of change to determine whether the function is linear. Graph each to check.

(a)

Table 1.11

Time (hours), x	Population (kg), y
0	0.09
1	0.12
2	0.16
3	0.22

Time (hours), x	Population (kg), y
4	0.29
5	0.39

(b)

Table 1.12

Age, x	Time (hours), y
20	50
30	47.5
40	45
50	42.5
60	40
70	37.5

☑ Solution

(a) Find the average rate of change for each pair of consecutive ordered pairs to see if it is constant, as shown in Table 1.13.

Table 1.13

Time (hours), x	Population (kg), y	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
0	0.09	$= \frac{0.12 - 0.09}{1 - 0} = \frac{0.03}{1} = 0.03$
1	0.12	$= 0.04$
2	0.16	$= 0.06$
3	0.22	$= 0.07$
4	0.29	$= 0.10$
5	0.39	

Since the average rate of change is not constant, we know that the function is not linear for the data in Table 1.11. Because the average rate of change is increasing as x increases, we say that the function is increasing at an increasing rate. In other words, not only is the population increasing over time, but it is also growing more rapidly as time passes. We also know that when we look at the graph of this function in Figure 1.27, the graph will be concave up.

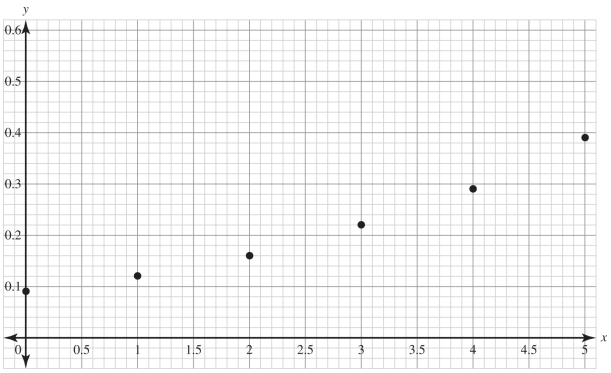


Figure 1.27

(b) Find the average rate of change for each pair of consecutive ordered pairs to see if it is constant, as shown in Table 1.14.

Table 1.14

Age, x	Time (hours), y	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
20	50	$= \frac{47.5 - 50}{30 - 20} = \frac{-2.5}{10} = -0.25$
30	47.5	$= -0.25$
40	45	$= -0.25$
50	42.5	$= -0.25$
60	40	$= -0.25$
70	37.5	

Since the average rate of change is constant, we know that the function is linear for the data in Table 1.12. When the data are graphed as in Figure 1.28, we can see that it resembles a straight line.

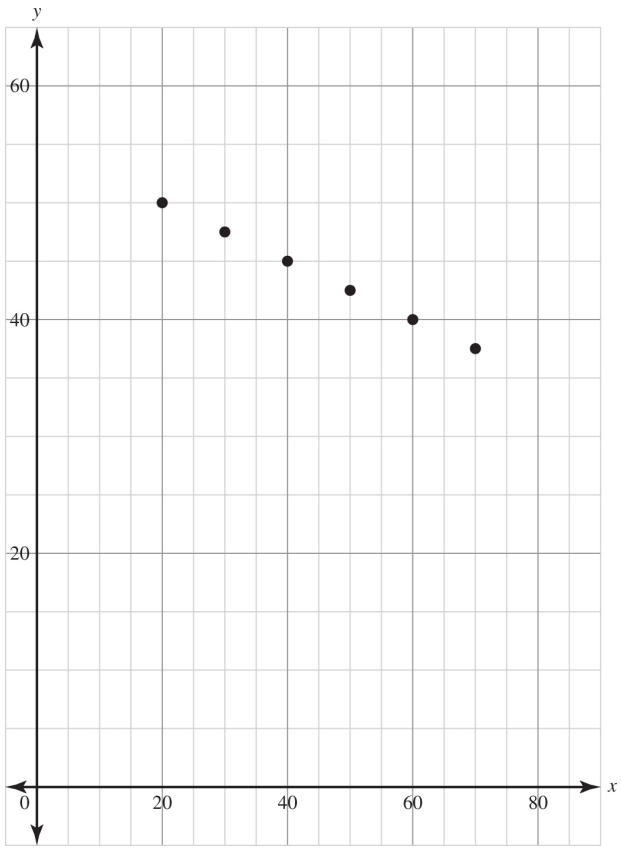


Figure 1.28

We can find the equation of the linear function using the point-slope formula $(y - y_1) = m(x - x_1)$ and then solving for y. Since the average rate

of change is −0.25, the slope is $m = -0.25$. Choose any point to substitute in for x_1 and y 1:

$$y - 50 = -0.25(x - 20)$$
$$y = -0.25x + 55$$

Since the average rate of change of the linear function in Table 1.12 is constant, the rate of change of the average rates of change of a linear function is zero. That is why the graph is neither concave up nor concave down.

Over any length interval, the average rate of change for a linear function is constant. The rate of change of the average rates of change of a linear function is zero.

1.4 Rates of Change in Quadratic Functions

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The domain of a quadratic function is the set of all real numbers.

As seen in the previous section, the rate of change of a linear function is constant. The rate of change of a quadratic function is not constant; there are no straight-line segments on a parabola, which is the graph of a quadratic function. To calculate the rate of change, determine the average rate of change by finding the slope of the segment that connects two points on the parabola. This segment is called the secant line.

Example

Consider the quadratic function $y = x^2$ and its graph shown in Figure 1.29.

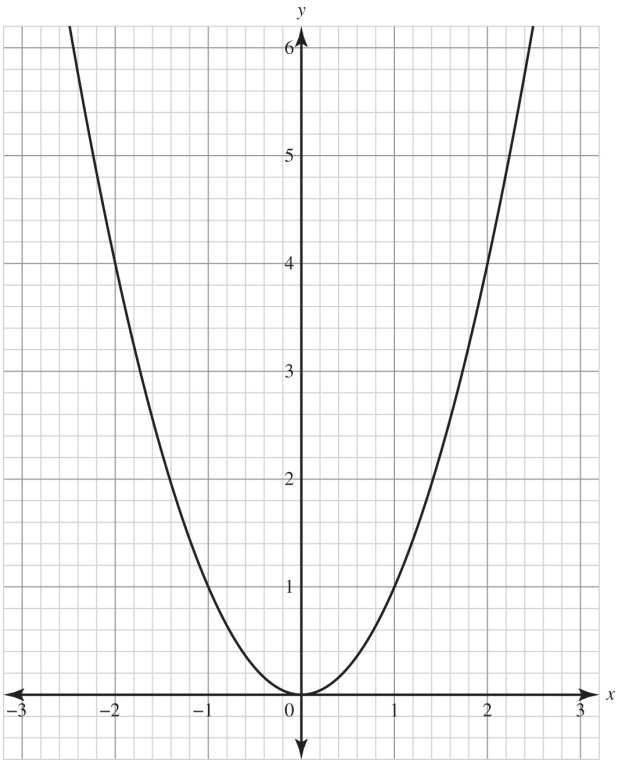


Figure 1.29

Table 1.15 shows the average rate of change between points that have consecutive, even-length input values.

Table 1.15

Points	Average Rate of Change = Slope of Secant Line
(0, 0) and (1, 1)	$= \frac{1-0}{1-0} = 1$
(1, 1) and (2, 4)	$= \frac{4-1}{2-1} = 3$
(2, 4) and (3, 9)	$= \frac{9-4}{3-2} = 5$

The graph in Figure 1.30 gives a visual representation of the slope of the secant lines.

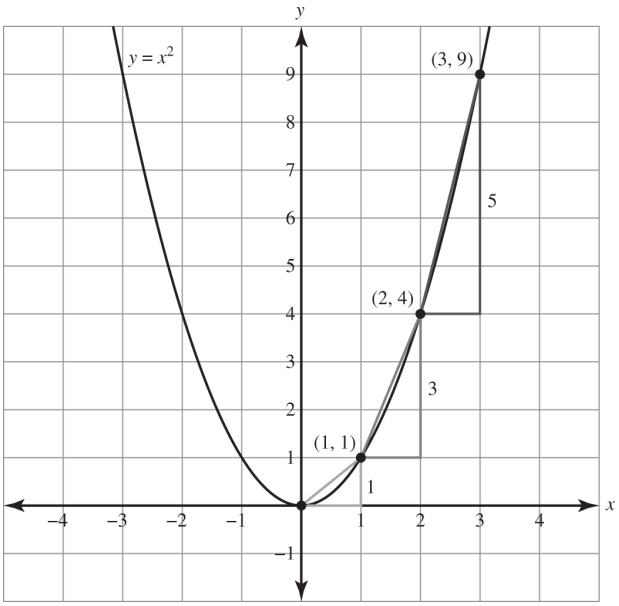


Figure 1.30

As x increases, we can see that the average rates of change are increasing, which coincides with the graph of the parabola being concave up. More specifically, since x increases by the equal interval of 1, we can see a more specific pattern emerging with the slopes of the secant lines. The rate of change of the average rates of change of this quadratic function is constant, specifically the constant 2. This will happen with any quadratic function.

Example

Which of the following functions represents a quadratic equation? State if the graph of the parabola would be concave up or concave down.

(a)

Table 1.16

x	$f(x)$
−3	−30
−2	−14

x	$f(x)$
−1	−4
0	0
1	−2
2	−10

(b)

Table 1.17

x	$g(x)$
−3	76
−2	55
−1	36
0	19
1	4
2	−9

(c)

Table 1.18

x	$h(x)$
−3	81
−2	16
−1	1
0	0

x	$h(x)$
1	1
2	16

☑ Solution

Since all the x-values in all three tables increase by 1, find the average rates of change between each pair of points.

(a)

Pair of Points from $f(x)$	Average Rate of Change
(−3, −30) and (−2, −14)	16
(−2, −14) and (−1, −4)	10
(−1, −4) and (0, 0)	4
(0, 0) and (1, −2)	−2
(1, −2) and (2, −10)	−8

The average rates of change are not constant, but the rate of change between each is a constant −6. Therefore, $f(x)$ is a quadratic function, and its parabola would be concave down.

(b)

Pair of Points from $g(x)$	Average Rate of Change
(−3, 76) and (−2, 55)	−21
(−2, 55) and (−1, 36)	−19
(−1, 36) and (0, 19)	−17

Pair of Points from $g(x)$	Average Rate of Change
(0, 19) and (1, 4)	−15
(1, 4) and (2, −9)	−13

The average rates of change are not constant, but the rate of change between each is a constant 2. Therefore, $g(x)$ is a quadratic function, and its parabola would be concave up.

(c)

Pair of Points from $h(x)$	Average Rate of Change
(−3, 81) and (−2, 16)	−65
(−2, 16) and (−1, 1)	−15
(−1, 1) and (0, 0)	−1
(0, 0) and (1, 1)	1
(1, 1) and (2, 16)	15

The average rates of change are not constant, and the rate of change between each is also not constant. Therefore, $h(x)$ is neither a linear nor a quadratic function.

Multiple-Choice Questions

1. Which of the following functions has a domain of $x > 0$?

(A) $f(x) = x^2$

(B) $f(x) = \sqrt{x}$

(C) $f(x) = \log x$

(D) $f(x) = e\ x$

2. Which of the following are functions?

I. $y - x = 17 + 2x$

II. $x^2 + y\ 2 = 64$

III. $x - y\ 2 = 81$

IV. $x^2 + y = 4$

(A) I only

(B) I and III only

(C) I and IV only

(D) I, II, III, and IV

3. Find the average rate of change of $f(x) = 3x^2 - 2$ from $x = 2$ to $x = 6$.

(A) 24

(B) 18

(C) $\frac{1}{18}$

(D) $\frac{1}{24}$

4. Which of the following linear functions is increasing?

(A) $s(t) = 4 - \frac{1}{2}t$

(B) $h(z) = 10$

(C) $g(x) = -3x + 2$

(D) $f(x) = -7 + 0.9x$

5. If a represents the zero of the function $f(x) = (x - 3)^2$, evaluate $g(a)$ if $g(x) = 10 - 2\ (x - 1)^3$.

(A) −6

(B) −3

(C) 1

(D) 3

6. As a pot of boiling water cools, the temperature of the water is recorded and placed in the table below, where time t is measured in minutes and temperature $F(t)$ is measured in degrees Fahrenheit. Using the data in the table, which of the following best approximates the rate at which the temperature of the water is changing at time $t = 3.5$?

t (minutes)	0	2	5	9	10
$F(t)$ (degrees Fahrenheit)	150.8	140	125.6	111.2	109.4

(A) −5.04°F/min
(B) −4.8°F/min
(C) −4.4°F/min
(D) −4.14°F/min

7. Given the table of values below, which of the following best describes the behavior of the graph of the function in the interval $1 \le x \le 6$?

x	1	2	3	4	5	6
y	11	6	−5	−22	−45	−74

(A) increasing, concave down
(B) increasing, concave up
(C) decreasing, concave down
(D) decreasing, concave up

8. Cailey loves to bake and decides to open her own cookie stand, named Cailey's Cookies. Her daily fixed costs are $120 a day, and each cookie costs $0.99 to bake. Which of the following represents a model that expresses the cost, C, of baking m cookies in a day?

(A) $C(m) = 0.99m + 120$
(B) $C(m) = 0.99x + 120$

(C) $C(m) = 120m + 0.99$

(D) $C(m) = 0.04125m + 5$

9. For the function $f(x) = x^2 - 4x + 1$, evaluate $f(x + h)$.

(A) $x^2 - 4x + 1 + h$

(B) $h^2 - 4h + 1$

(C) $x^2 + h^2 + 2xh - 4x - 4h + 1$

(D) $x^2 + h^2 - 4x - 4h + 1$

10. Which of the following functions has the domain $(-\infty, \infty)$, $x \neq -9$, $x \neq 4$?

(A) $f(x) = \sqrt{x - 36}$

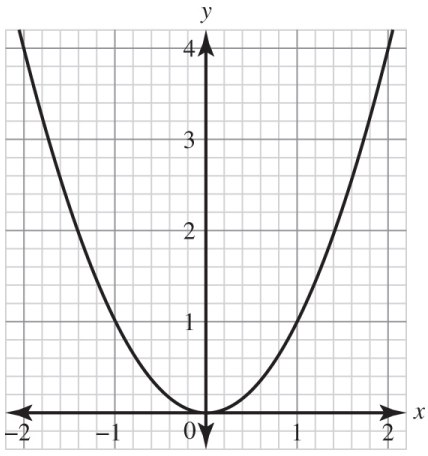
(B) $g(x) = \dfrac{x - 3}{x^2 + 5x - 36}$

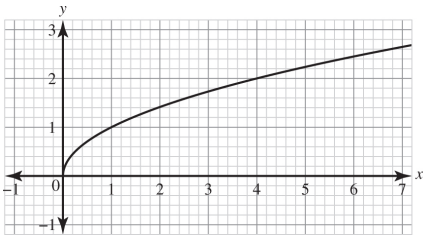
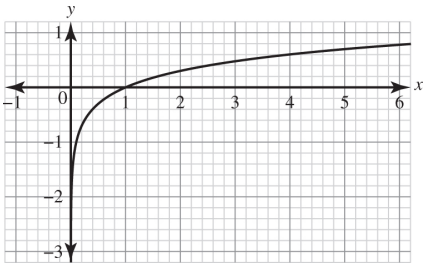
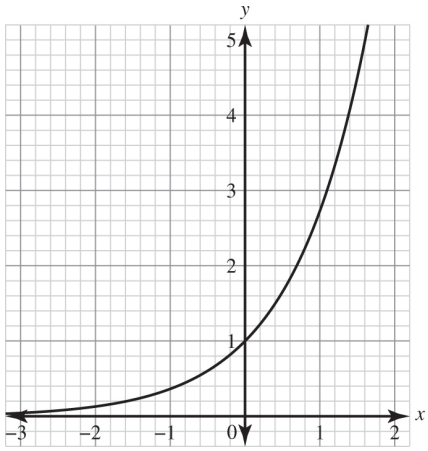
(C) $h(x) = \dfrac{x}{(x + 4)(x - 9)}$

(D) $i(x) = x^2 + 5x - 36$

Answer Explanations

1. **(C)** The domain of each of the functions can be seen from their graphs.

$f(x)$	Graph	Domain
x^2		All real numbers

$f(x)$	Graph	Domain
$\sqrt{x}$		$x \geq 0$
$\log x$		$x > 0$
e^x		All real numbers

2. **(C)** For an equation to be a function, every x can correspond to one and only one y. This can be verified graphically by seeing if the graph passes the vertical line test or algebraically by solving for y and seeing if each y has only one solution. The only issues are II and III since the y is squared. When isolating the y, we extract the square root. This would give two solutions, one positive and one negative.

I. $y - x = 17 + 2x$	II. $x^2 + y\ 2 = 64$	III. $x - y\ 2 = 81$	IV. $x^2 + y = 4$
$y = 3x + 17$	$y^2 = 64 - x^2$ $y = \pm\sqrt{64 - x^2}$	$-y^2 = 81 - x$ $y^2 = -81 + x$ $y = \pm\sqrt{-81 + x}$	$y = 4 - x\ 2$

3. (A) To find the average rate of change of a quadratic function, calculate the slope of the secant line of the endpoints provided. At $x = 2$, $y = 10$, and at $x = 6$, $y = 106$. The average rate of change = slope of secant line $= \frac{\Delta y}{\Delta x} = \frac{106 - 10}{6 - 2} = 24$.

4. (D) To determine if a linear function is increasing, decreasing, or constant, determine the slope. A linear function is increasing if the slope is positive, decreasing if the slope is negative, and constant if the slope is zero. All the choices are written in the form $y = mx + b$, where m represents the slope. The only linear function with a positive slope is $f(x) = -7 + 0.9x$.

5. (A) First find the value of a. To find a zero, set the equation equal to zero and solve for the independent variable:

$$\begin{aligned} 0 &= (x-3)^2 \\ \sqrt{0} &= \sqrt{(x-3)^2} \\ 0 &= x - 3 \\ x &= 3 = a \end{aligned}$$

To evaluate $g(a)$, substitute $a = 3$ for x in the function $g(x)$:

$$g(a) = g(3) = 10 - 2(3-1)^3 = 10 - 2(2)^3 = -6$$

6. (B) The rate of change at a point can be approximated by the average rates of change of the function over small intervals containing the point. The smallest interval containing $t = 3.5$ in the table is from $t = 2$ to $t = 5$. Calculate the average rate of change:

$$\frac{125.6 - 140}{5 - 2} = \frac{-14.4}{3} = -4.8^\circ\text{F/min}$$

7. (C) Since the y-values are getting smaller as x increases over the interval, the function is decreasing. One way to describe the concavity of the function is to calculate and interpret the average rate of change between intervals of equal length. Since the given table has x increase by 1 each time, calculate the average rate of change between points as shown in the table.

Points Used	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
(1, 11) and (2, 6)	−5
(2, 6) and (3, −5)	−11
(3, −5) and (4, −22)	−17
(4, −22) and (5, −45)	−23
(5, −45) and (6, −74)	−29

The average rate of change is not constant, so it is not linear. The rate of change of the average rate of change is a constant −6, so the function is quadratic. Since the average rates of change are decreasing, the parabola is concave down over the interval.

8. (A) Since we are asked to write a model for only one day, the fixed costs of $120 occur only once and represent the initial cost even if zero cookies are made. The total cost depends on how many cookies are baked. The number of cookies baked is the independent variable, m. The $0.99 represents the cost per cookie and reflects the slope of the linear model. Since it is a constant rate of change, the cost to bake the cookie is not going to change. Using the slope-intercept form of a line, the linear cost model is written as $C(m) = 0.99m + 120$.

9. (C) To evaluate $f(x + h)$, substitute $(x + h)$ for x in the given function $f(x) = x^2 - 4x + 1$:

$$\begin{aligned} f(x+h) &= (x+h)^2 - 4(x+h) + 1 \\ &= (x+h)(x+h) - 4x - 4h + 1 \\ &= x^2 + xh + xh + h^2 - 4x - 4h + 1 \\ &= x^2 + 2xh + h^2 - 4x - 4h + 1 \end{aligned}$$

10. (B) The domain of a function is the set of input values for which the function is defined and has a corresponding output value. In Choice (A), a radical cannot be a negative number. So the domain for $f(x)$ is $x \geq 36$. For functions like Choices (B) and (C), there cannot be a zero in the denominator. Set each denominator equal to zero and solve for x. Then

restrict these values from the domain. In Choice (B), $x^2 + 5x - 36 = 0$, by factoring $(x + 9)(x - 4) = 0$, and therefore $x \neq -9$, $x \neq 4$. In Choice (C), $(x + 4)(x - 9) = 0$, which means $x \neq -4$, $x \neq 9$. Choice (D) is a quadratic equation, so its domain is all real numbers.

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2

Polynomial Functions

Learning Objectives

In this chapter, you will learn:

- ➜ The different properties of polynomial functions
- ➜ How to graph polynomial functions
- ➜ How to calculate the rate of change of different functions
- ➜ How to write equivalent forms of polynomial functions
- ➜ How to solve polynomial inequalities

A polynomial function, p, is a function that can be in the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$, where $a_n \neq 0$ and n is a positive integer called the degree of the polynomial. Each member of the sum is a term of the polynomial. The number constants a_n, a_{n-1}, a_{n-2}, . . ., a_1, a_0 are coefficients. The leading term is $a_n x^n$, the leading coefficient is a_n, and the constant term is a_0.

Polynomial functions should be written in descending order, starting with the term of the highest power. Any term missing in the polynomial means that 0 is the coefficient of that term. A constant is considered a polynomial function of degree 0.

Example

Which of the following functions are considered polynomial functions?

1. $f(x) = x^5 + \sqrt{x}$
2. $g(x) = x^3 - \frac{1}{x}$
3. $h(x) = 7x^4 - \frac{1}{2}x^3 + x - 10$

☑ Solution

1. This is not a polynomial function since it contains $\sqrt{x} = x^{1/2}$.
2. This is not a polynomial function since it contains $\frac{1}{x} = x^{-1}$.
3. This is a polynomial function since each of the exponents in $h(x) = 7x^4 - \frac{1}{2}x^3 + x - 10$ is a positive, whole number.

2.1 Polynomial Functions and Rates of Change

In Chapter 1, calculating the rate of change helped to distinguish between linear and quadratic functions and helped discover the concavity of quadratic functions. Since linear functions (degree 1) and quadratic functions (degree 2) are both types of polynomial functions, we will now discover how the rate of change can be applied to polynomial functions of all degrees.

Relative Extrema

A relative extremum (also known as a local extremum) means the existence of a relative maximum or minimum point on a graph. This is a point that looks like it is either the top or bottom of a hill. More formally, where a polynomial function switches between increasing and decreasing, the polynomial function will have a local (relative) maximum or minimum output value.

This behavior can easily be identified in our basic quadratic functions, $f(x) = x^2$ and $g(x) = -x^2$.

For $f(x) = x^2$, shown in Figure 2.1, the graph is decreasing when $x < 0$ and increasing when $x > 0$. At $x = 0$, there is a local minimum.

For $g(x) = -x^2$, shown in Figure 2.2, the graph is increasing when $x < 0$ and decreasing when $x > 0$. At $x = 0$, there is a local maximum.

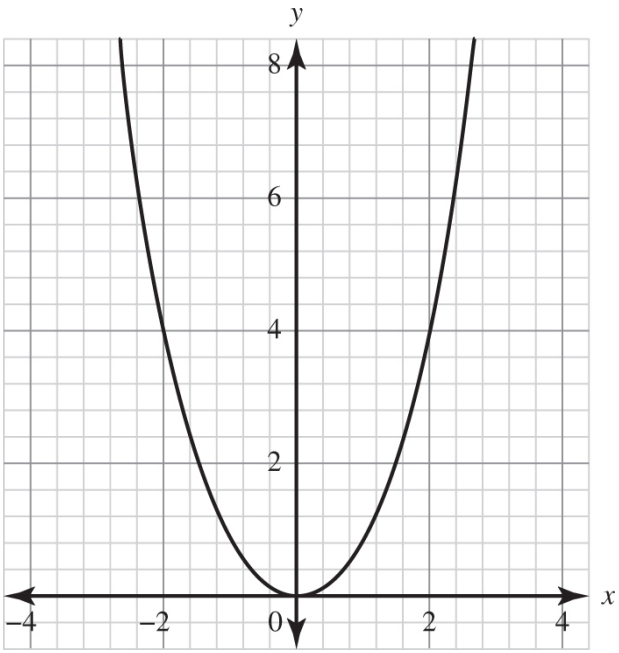


Figure 2.1

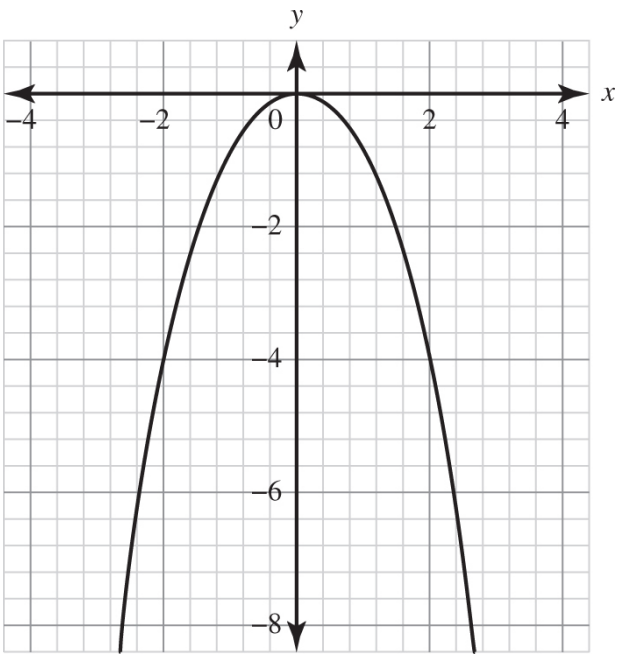


Figure 2.2

If a polynomial function has a restricted domain, at the included endpoints the polynomial will also have a local extremum value.

Consider the linear function $f(x) = x + 5$ over the domain $[-2, 4]$. This linear function is of degree 1 and is considered a polynomial. By restricting the domain, the graph of $f(x)$ is shown in Figure 2.3.

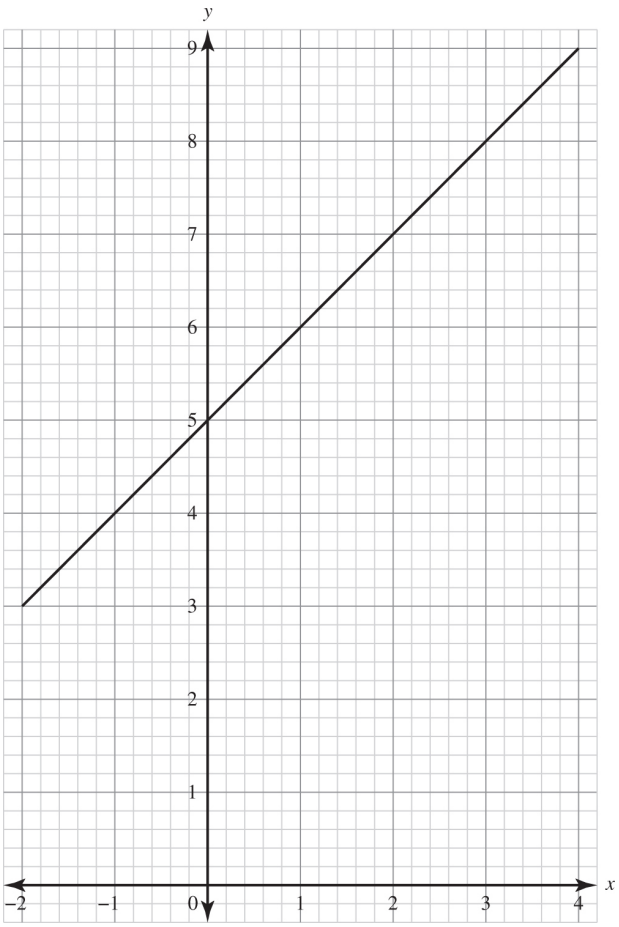


Figure 2.3

At the endpoint $(-2, 3)$, there is a local minimum. At the endpoint $(4, 9)$, there is a local maximum.

This leads to two important theorems, the extreme value theorem and the local extrema theorem, as defined on the right.

Extreme Value Theorem

If a polynomial function f is on a closed interval $[a, b]$, then f has both a maximum and a minimum value on $[a, b]$.

Local Extrema Theorem

A polynomial function of degree n has at most $n - 1$ relative maxima/minima.

Example

State the number of possible relative maxima/minima of $f(x) = x^4 - x^3 - 5x^2 + 1$. Confirm your answer graphically.

Solution

The degree of $f(x)$ is 4, meaning there are at most $4 - 1 = 3$ relative extrema. It can be expected that there will be 3 or less possible relative maximum or relative minimum points on the graph of $f(x)$.

Figure 2.4 is the graph of $f(x) = x^4 - x^3 - 5x^2 + 1$. The graph has two relative minima at $x = -1.25$ and $x = 2$ and one relative maximum at $x = 0$.

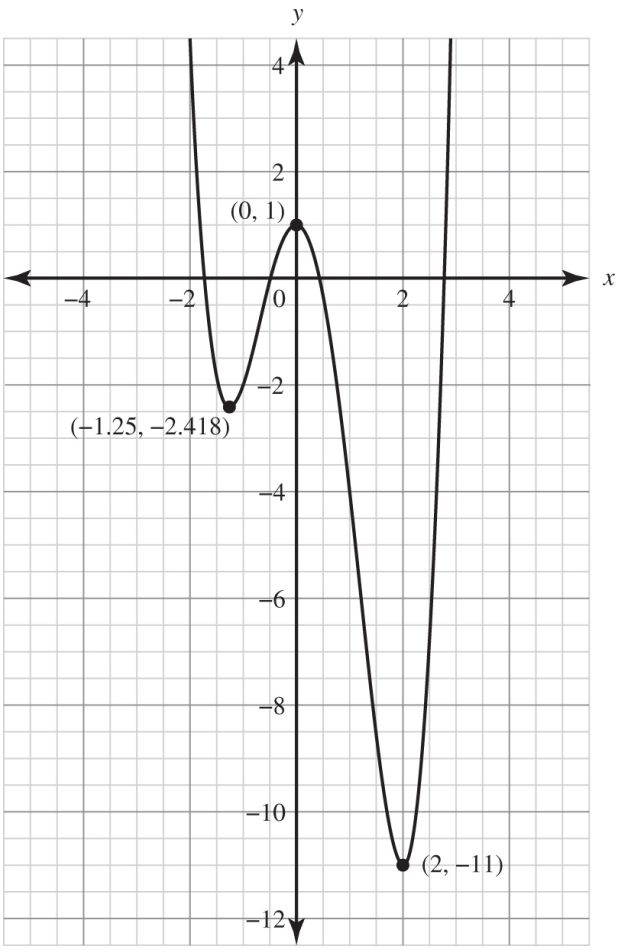


Figure 2.4

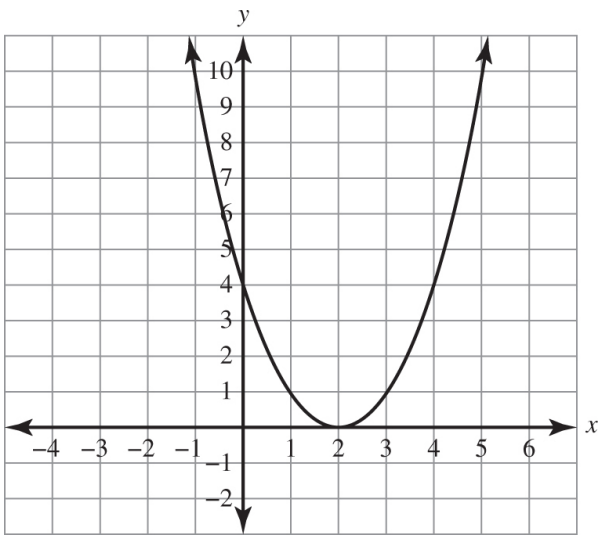
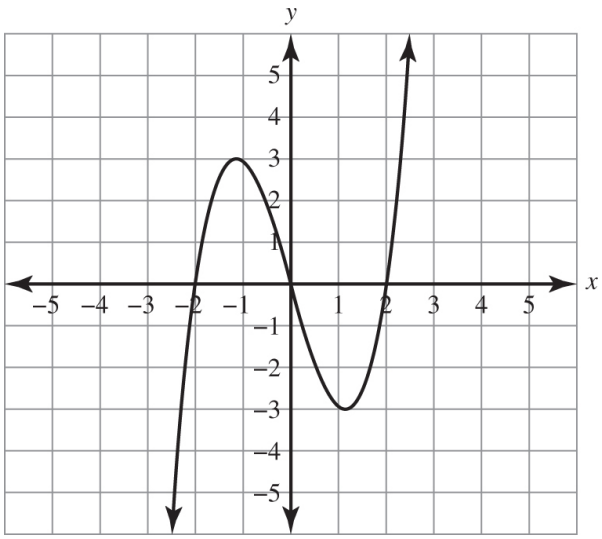
Absolute Extrema

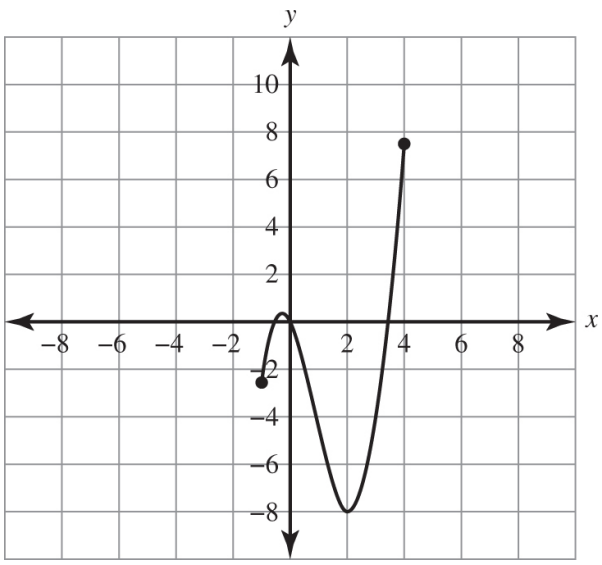
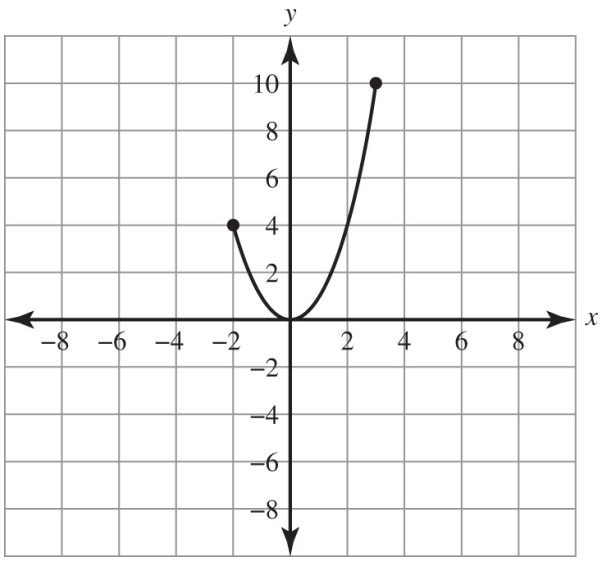
Over a closed interval, the greatest of all the local maxima is called the global, or absolute, maximum. Likewise, the least of all local minima is called the global, or absolute, minimum.

The graphs in Table 2.1 discuss the existence of absolute extrema.

Absolute extrema can occur at either relative extrema or at the endpoints of the closed interval.

Table 2.1

Graph	Absolute Extrema
 1	The graph has an absolute minimum at $x = 2$ but no absolute maxima since the domain is not a closed interval.
 2	The graph has neither absolute maxima nor absolute minima since the domain is not a closed interval. There is a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

Graph	Absolute Extrema
 3	The graph has an absolute minimum at $x = 2$, which is also a relative minimum, and an absolute maximum at $x = 4$, which is also an endpoint.
 4	The graph has an absolute minimum at $x = 0$, which is also a relative minimum, and an absolute maximum at $x = 3$, which is also an endpoint.

In graphs 2 and 3, the polynomials have zeros where the graphs intersect the x-axis. It can be observed that between every two distinct real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or local minimum. In graph 2, there are zeros at $x = -2$ and $x = 0$ and a relative maximum at $x = -1$. There is also a set of zeros at $x = 0$ and $x = 2$ and a relative minimum at $x = 1$.

Graphs 1 and 4 are parabolas, meaning they are a type of quadratic equation. Both have one relative extremum. When the domain is all real numbers as it is in graph 1, the one relative extremum is also the absolute extremum.

In all the graphs, the polynomial functions are continuous, meaning no breaks, and have smooth curves. In graphs 1 and 4, we see that the graphs are always concave up. However, in graphs 2 and 3, the graphs go from concave down to concave up. The point where the concavity changes is known as the point of inflection. For graph 2, the point of inflection is at $x = 0$, and for graph 3, the point of inflection is at $x = 1$.

Polynomial functions of an even degree have either a global maximum or a global minimum.

Point of Inflection Theorem

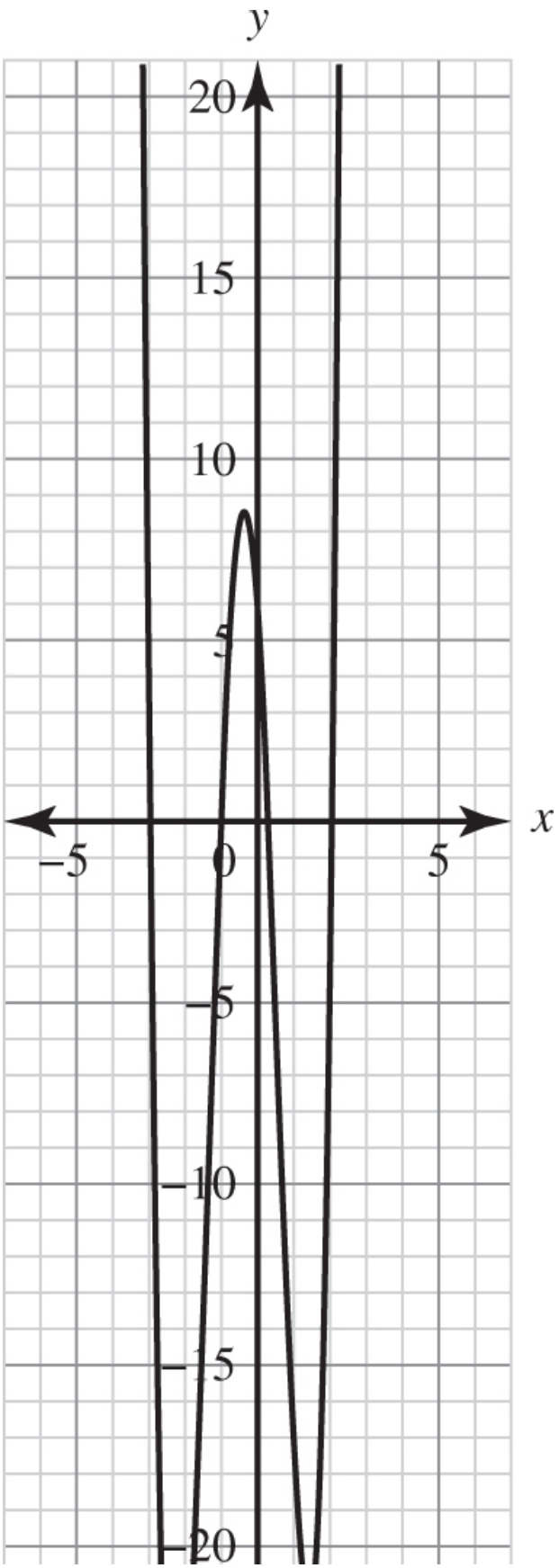
The graph of a polynomial function of degree n, with $n \geq 2$, has at most $n - 2$ points of inflection. The graph of a polynomial function of odd degree has at least 1 point of inflection.

Example

State the number of possible points of inflection of $p(x) = 3x^4 + 5x^3 - 17x^2 - 14x + 6$. Use a graph to estimate their location.

Solution

The degree of the polynomial is 4. There will be at most $4 - 2 = 2$ points of inflection, as shown in Figure 2.5.


y
20
15
10
5
x
−5
0
5
−5
−10
−15
−20

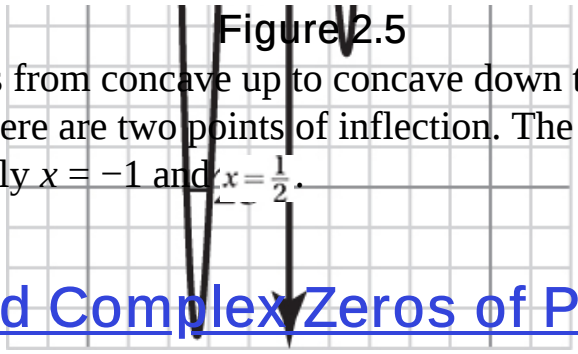


Figure 2.5

The graph changes from concave up to concave down to concave up. With the two changes there are two points of inflection. The points of inflection are at approximately $x = -1$ and $x = \frac{1}{2}$.

2.2 Real and Complex Zeros of Polynomial Functions

There are different ways to find the zeros of a polynomial function. Graphically, the zeros are located at the x-intercepts. Algebraically, the equation is set equal to 0 and solved for the independent variable. As the degree of the polynomial function increases and becomes larger than 2, it is common that the equation will have to be factored more than once or a graphing calculator will have to be used to locate the x-intercepts.

The degree of the polynomial states how many zeros the function has, both real and imaginary. For the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$, where $a_n \neq 0$, there will be n complex zeros. A complex number is of the form $a + bi$, where $i = \sqrt{-1}$ and where a and b are real numbers. The real component of the number is a, and bi is the imaginary component. While a polynomial function of degree n will have exactly n complex zeros, it doesn't necessarily mean that it will have exactly n real zeros. This means that the graph of a polynomial function will not always have the same number of x-intercepts as its degree. This may be due to repeated zeros or imaginary zeros.

› Example

Find the zeros of each of the polynomial functions. Confirm your answer graphically.

1. $f(x) = x^3 - 3x^2 - 4x + 12$
2. $g(x) = x^4 - 5x^2 - 36$

✓ Solution

To algebraically find the zeros, set each equation equal to 0 and factor to solve for x.

1. Factor this equation by grouping. Group together the first two terms and the last two terms. Then factor out their respective greatest common factors:

$$\begin{aligned} x^3 - 3x^2 - 4x + 12 &= 0 \\ (x^3 - 3x^2) + (-4x + 12) &= 0 \\ x^2(x - 3) - 4(x - 3) &= 0 \\ (x^2 - 4)(x - 3) &= 0 \end{aligned}$$

   The first factor can be factored again since it is a difference of two perfect squares:

$$(x + 2)(x - 2)(x - 3) = 0$$

   Set each factor equal to zero and solve for x:

$$x = -2, x = 2, x = 3$$

   There are three real zeros of this polynomial function.

   Consider the graph of $f(x)$ in Figure 2.6.

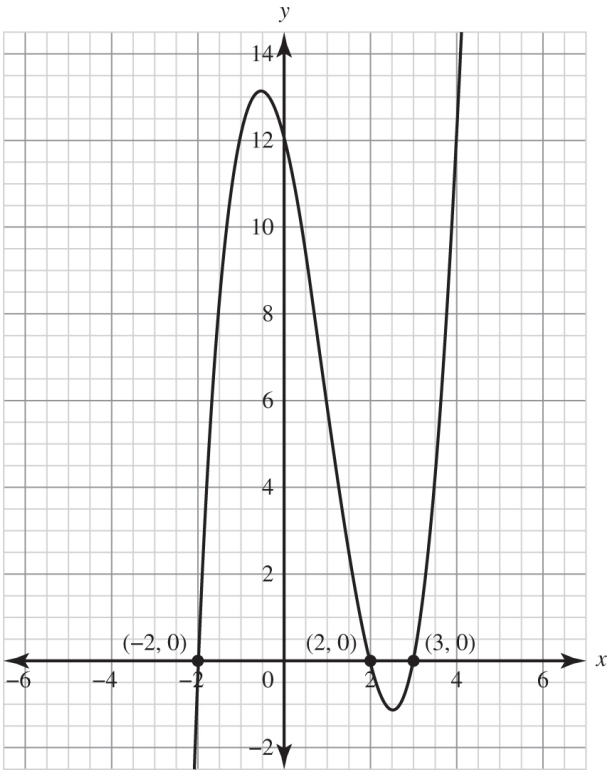


Figure 2.6

There are three *x*-intercepts, (−2, 0), (2, 0), and (3, 0). As expected, the *x*-coordinates match the zeros found algebraically.

This was a third-degree polynomial that had three real zeros and three *x*-intercepts.

2. Factor this equation using product-sum:

$$x^4 - 5x^2 - 36 = 0$$
$$(x^2 - 9)(x^2 + 4) = 0$$

The first factor can be factored again since it is a difference of two perfect squares. The second factor is a sum of two perfect squares and cannot be factored with real numbers:

$$(x + 3)(x - 3)(x^2 + 4) = 0$$

Set each factor equal to zero and solve for *x*:

$$x = -3, x = 3, x^2 = -4$$

To solve for x for the last factor, extract the square root:

$$\sqrt{x^2} = \sqrt{-4}$$

$$x = \pm\sqrt{-4}$$

Simplify the square root using imaginary numbers:

$$x = 2i, x = -2i$$

There are four complex zeros: $x = 3$, $x = -3$, $x = 2i$, $x = -2i$. There are two real zeros and two imaginary zeros.

Consider the graph of $g(x)$ in Figure 2.7.

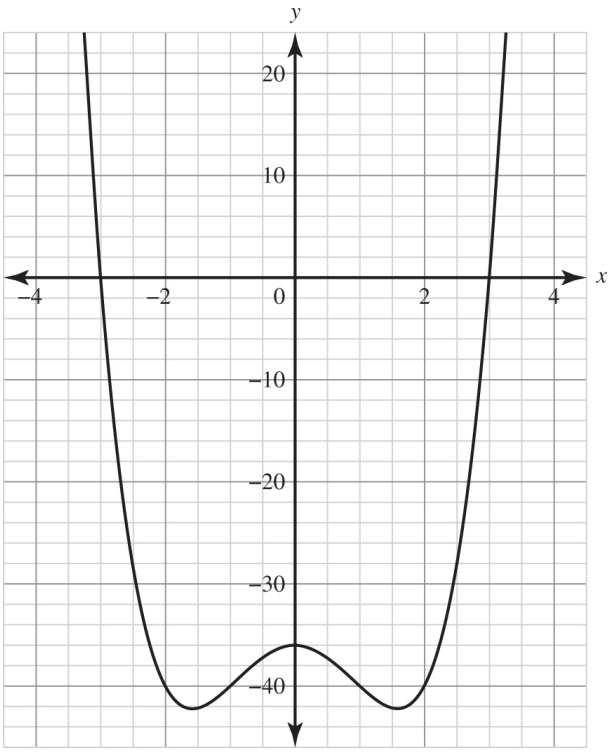


Figure 2.7

There are two x-intercepts, $(-3, 0)$ and $(3, 0)$. Only the real zeros are x-intercepts for the graph. Since the coordinate plane graphs only real numbers, the imaginary zeros do not appear on the graph.

A few key properties were demonstrated in the previous examples.

Key Properties of the Polynomial Function $p(x)$

1. If a is a complex number and $p(a) = 0$, then a is called *a* zero of *p*, or a root of the polynomial function p.
2. If a is a real number, then $(x - a)$ is a linear factor of *p* if and only if a is a zero of *p*.
3. If a is a real zero of a polynomial function *p*, the graph of $y = p(x)$ has an *x*-intercept at the point $(a, 0)$.
4. If $a + bi$ is a nonreal zero of a polynomial *p*, its conjugate $a - bi$ is also a zero of *p*.

Example

Find the zeros of each function.

1. $p(x) = x^4 - 25$
2. $q(x) = x^3 - 2x^2 + 5x$

Solution

Set each equation equal to 0, and factor each polynomial completely.

1.

$$x^4 - 25 = 0$$
$$(x^2 - 5)(x^2 + 5) = 0$$
$$(x - \sqrt{5})(x + \sqrt{5})(x - i\sqrt{5})(x + i\sqrt{5}) = 0$$

The zeros of the polynomial are $x = \sqrt{5}, -\sqrt{5}, i\sqrt{5}, -i\sqrt{5}$. The irrational zeros occur in conjugate pairs, as do the imaginary zeros.

2.

$$x^3 - 2x^2 + 5x = 0$$
$$x(x^2 - 2x + 5) = 0$$
$$x = 0, x^2 - 2x + 5 = 0$$

Use the quadratic formula to find the other two zeros.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

The zeros of the polynomial are $x = 0, 1 + 2i, 1 - 2i$. There is one real zero, and the two imaginary zeros occur in conjugate pairs.

Example

Find a polynomial function $f(x)$ with real coefficients of degree 4 that has the zeros -2, 3, and $-1 + i$ and whose leading coefficient is 1.

Solution

Since it is given that $f(x)$ is degree 4, there should be exactly four complex zeros. Two of the three given zeros are real, and the third zero is $-1 + i$. That means that the fourth zero of $f(x)$ is its conjugate pair, $-1 - i$. Now that all four zeros have been found, their linear factors can be written, as shown in Table 2.2.

Table 2.2

Zero	$x = -2$	$x = 3$	$x = -1 + i$	$x = -1 - i$
Factor	$(x + 2)$	$(x - 3)$	$(x - (-1 + i))$	$(x - (-1 - i))$

Since the leading coefficient was given as 1, to find the polynomial, multiply the factors together:

$$f(x) = (x + 2)(x - 3)(x - (-1 + i))(x - (-1 - i))$$
$$f(x) = (x^2 - 3x + 2x - 6)(x^2 - (-1 - i)x - (-1 + i)x + (-1 + i)(-1 - i))$$
$$f(x) = (x^2 - x - 6)(x^2 + 1x + ix + 1x - ix + 1 + i - i - i^2)$$
$$f(x) = (x^2 - x - 6)(x^2 + 2x + 2)$$
$$f(x) = x^4 + x^3 - 6x^2 - 14x - 12$$

Counting Repeated Zeros

There is no guarantee that linear factors of a polynomial are all different. For example, $p(x) = x^2 - 8x + 16 = (x - 4)(x - 4) = (x - 4)^2$. Since the degree is 2, one might assume there would be two zeros. However, $p(x)$ has two equal zeros, $x = 4$. Since $x - 4$ is repeated two times when $p(x)$ is written as a product of linear factors, it is said that "4 is a zero of multiplicity two."

Repeated zeros have an interesting relationship when the polynomial is graphed, as shown in Figure 2.8.

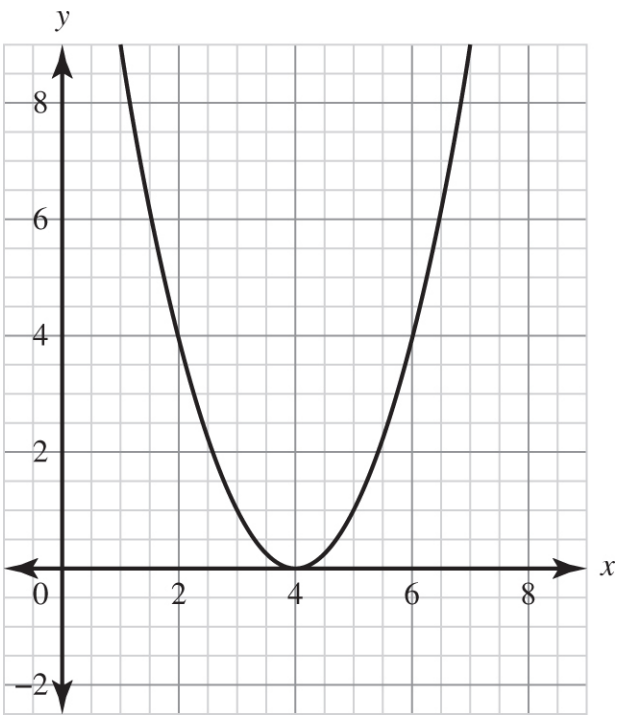


Figure 2.8

For the graph of $p(x) = (x - 4)^2$, at the zero $x = 4$, the graph has an x-intercept at (4, 0) but does not pass through the x-axis. Instead, the graph seems to bounce off. This is known as the graph being tangent to the x-axis at $x = 4$. It is also important to note that the x-values close to $x = 4$, both on the left and right side, have corresponding y-values that are positive.

When polynomial functions are written as a product of their linear factors, the sum of the multiplicities of each factor represents the degree of the polynomial.

Example

Given: $f(x) = (1 - x)^3 (4 + x)^2$

(a) Find the degree of the polynomial.

(b) Discuss the behavior of the graph of $f(x)$ at the x-intercepts.

Solution

(a) When all the factors of the polynomial function are multiplied together, the term with the greatest power will be $(x^3)(x^2) = x^5$. So the degree of $f(x)$ is 5.

The sum of the multiplicities of the factors is 3 + 2 = 5, confirming the degree of $f(x)$. Since $f(x)$ is a fifth-degree polynomial, it has a total of five complex zeros. Specifically, 1 is a zero of multiplicity 3, and −4 is a zero of multiplicity 2.

(b) To see the behavior of the graph at both an even and odd multiplicity, consider the graph of $f(x)$ shown in Figure 2.9.

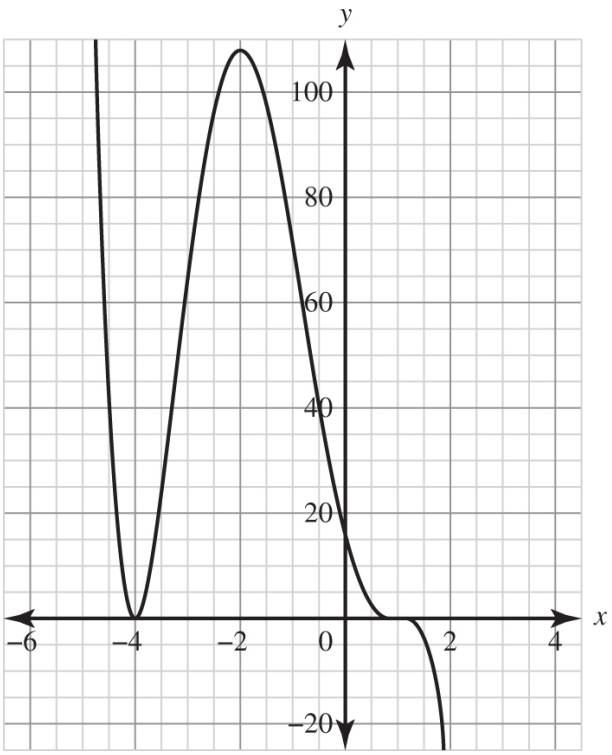


Figure 2.9

The behaviors at each of the x-intercepts are different. At (1, 0) the graph crosses the x-axis, but at (−4, 0) the graph is tangent to the x-axis.

Remember

When counting possible zeros from the graph of a function, we cannot assume that an x-intercept corresponds to exactly one zero. The following generalizations exist:

1. A polynomial function of degree n has exactly n complex zeros, provided that repeated zeros are counted the number of times they occur as factors when the function is written as a product of linear factors.
2. If the graph of a polynomial function $y = f(x)$ crosses the x-axis at $(k, 0)$, then k occurs as a zero an odd number of times.
3. If the graph of a polynomial function $y = f(x)$ is tangent to the x-axis at $(k, 0)$, then k occurs as a zero an even number of times.

Example

Find a polynomial function f for which 2 is a zero of multiplicity 2 and $\sqrt{5}$ is also a zero.

Solution

Since 2 is a zero of multiplicity 2 for function f, $(x - 2)^2$ is a factor of the function. Since $\sqrt{5}$ is a zero, then its conjugate, $-\sqrt{5}$, is also a zero. Therefore, $(x - \sqrt{5})$ and $(x + \sqrt{5})$ are also factors of function f.

Multiply to find one possible polynomial function for $f(x)$:

$$f(x) = (x-2)^2(x-\sqrt{5})(x+\sqrt{5}) = (x^2-4x+4)(x^2-5)$$
$$f(x) = x^4 - 4x^3 - x^2 + 20x - 20$$

Example

Find a polynomial function f that would match the graph in Figure 2.10.

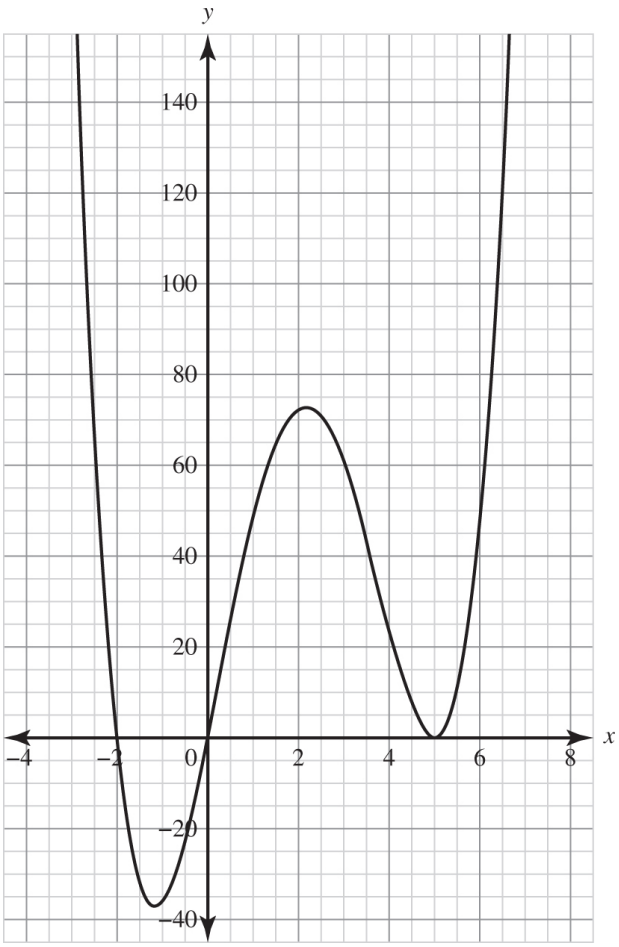


Figure 2.10

Solution

Table 2.3 shows a way to organize the given and needed information.

Table 2.3

Coordinates of the x-intercept	$(-2, 0)$	$(0, 0)$	$(5, 0)$
Zero	$x = -2$	$x = 0$	$x = 5$
Graph behavior at the x-intercept	Crosses x-axis	Crosses x-axis	Tangent to the x-axis
Multiplicity	Odd	Odd	Even
Possible factor	$(x + 2)$	x	$(x - 5)^2$

Assuming the leading coefficient is 1, one possible function is $f(x) = x(x + 2)(x - 5)^2$.

Line and Origin Symmetry

When working with graphs of functions, it is helpful to know whether the function has any special symmetry.

A graph has line symmetry if the graph can be mapped onto itself by a reflection over a line. Figure 2.11 shows a graph with vertical line symmetry.

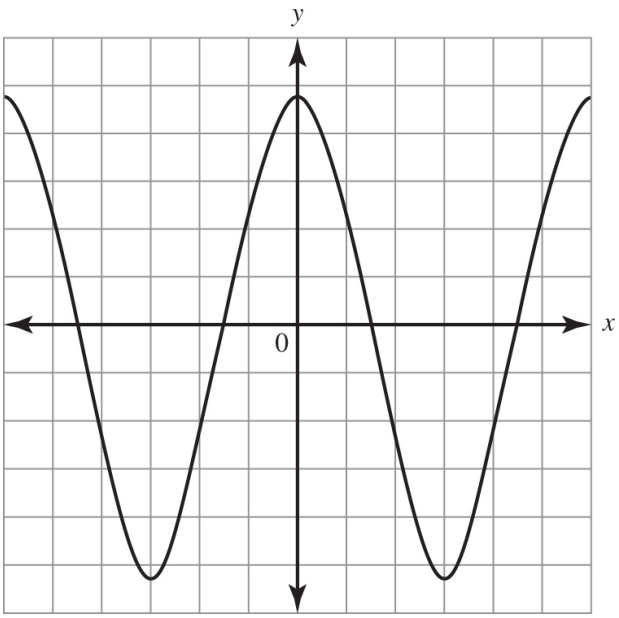


Figure 2.11

A graph is symmetric to the origin if, after it is reflected over both the *x*- and *y*-axes, in either order, the new graph coincides with the original graph. The graph in Figure 2.12 is symmetric to the origin.

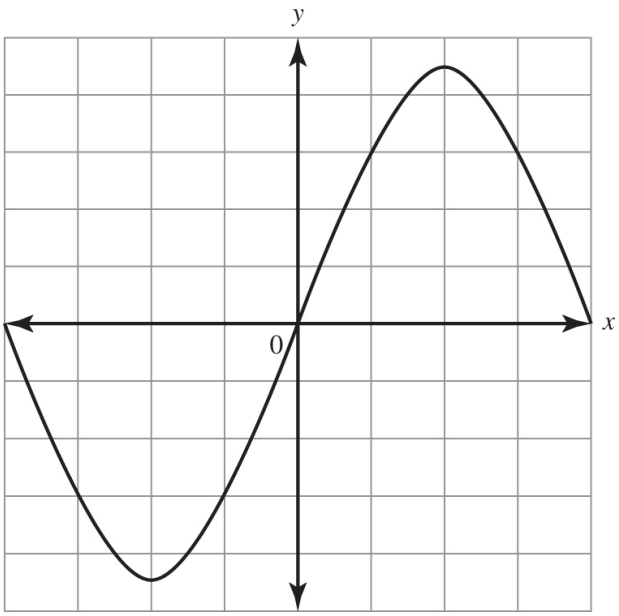


Figure 2.12

Even and Odd Functions

Functions that have y-axis symmetry or origin symmetry are given special names.

A function f is an even function if its equation remains unchanged when x is replaced with $-x$, that is, $f(-x) = f(x)$ for all x. Even functions are symmetric to the y-axis. For example, the function $f(x) = x^2$ is an even function since replacing x with $-x$ gives $f(-x) = (-x)^2 = x^2 = f(x)$. The graph in Figure 2.11 is an even function since it has y-axis symmetry.

A function f is an odd function if replacing x with $-x$ changes the sign of each term of the equation to its opposite, that is, when $f(-x) = -f(x)$ for all x. The graph of an odd function is symmetric to the origin. For example, the function $f(x) = x^3 - x$ is an odd function since replacing x with $-x$ gives $f(-x) = (-x)^3 - (-x) = -x^3 + x = -f(x)$. The graph in Figure 2.12 is an odd function since it has origin symmetry.

Example

The function $f(x)$ over the domain $-10 \le x \le 10$ is represented for selected values in Table 2.4. Given that $f(x)$ is an even function, complete the rest of the table.

Table 2.4

x	−10	−4	−1	0	1	4	10
$f(x)$	18		2	0		−3	

Solution

Since the function is even, the graph is symmetric with respect to the y-axis, meaning that the y-value is the same for every positive and negative x-value. This is shown in Table 2.5.

Table 2.5

x	−10	−4	−1	0	1	4	10
$f(x)$	18	−3	2	0	2	−3	18

2.3 Polynomial Functions and End Behavior

The graphs of all polynomial functions are continuous because there are no values of x for which the function would be undefined. That means that there are no breaks, holes, jumps, or asymptotes in the graph.

The discussion of end behavior of the graph of a polynomial function is an interesting one since the domain of a polynomial function is all real numbers, meaning the domain does not end. To really examine the end behavior, the concept of a limit must be introduced.

Definition of a Limit

Given a function $f(x)$, the limit of $f(x)$ as x approaches c is a real number L if $f(x)$ can be made arbitrarily close to L by having x approach close to c but not be equal to c. Table 2.6 shows limit notation and how to read it.

Table 2.6

Notation	Read As
$\lim_{x \to c} f(x) = L$	The limit as x approaches c of $f(x)$ equals L

If the limit equals a value L, it is said that the limit exists. It is also understood that the values of $f(x)$ approach L as x approaches c from both the left and the right.

Our focus is on end behavior, so our limit won't be approaching a value. Instead for x to go all the way to the end on the left, it will be going toward negative infinity, and on the right, it will be going toward positive infinity.

The symbol ∞ is used to represent infinity. Infinity is the idea of something unlimited, without bounds. Infinity can appear in limits in two ways: as what x is approaching or as what the function is approaching.

To test for end behavior, first determine whether the leading coefficient of the polynomial is either positive or negative. Then determine if the degree is either even or odd. This leads to four possible scenarios for the polynomial $p(x)$, as explained in Table 2.7.

Table 2.7

Leading Coefficient	Degree	End Behavior	Limit Notation
Positive	Even	Graph rises on the left and right	$\lim_{x\to-\infty} p(x) = +\infty$ $\lim_{x\to\infty} p(x) = +\infty$
Positive	Odd	Graph falls on the left and rises on the right	$\lim_{x\to-\infty} p(x) = -\infty$ $\lim_{x\to\infty} p(x) = +\infty$
Negative	Even	Graph falls on the left and right	$\lim_{x\to-\infty} p(x) = -\infty$ $\lim_{x\to\infty} p(x) = -\infty$

Leading Coefficient	Degree	End Behavior	Limit Notation
Negative	Odd	Graph rises on the left and falls on the right	$\lim_{x \to -\infty} p(x) = +\infty$ $\lim_{x \to \infty} p(x) = -\infty$

This behavior can be easily identified in two basic polynomial functions, $f(x) = x^2$ and $g(x) = x^3$. For $f(x) = x^2$, the leading coefficient (1) is positive, and the degree (2) is even. In Figure 2.13, it can be verified that the graph rises on the left, $\lim_{x \to -\infty} f(x) = +\infty$, and rises on the right, $\lim_{x \to \infty} f(x) = +\infty$.

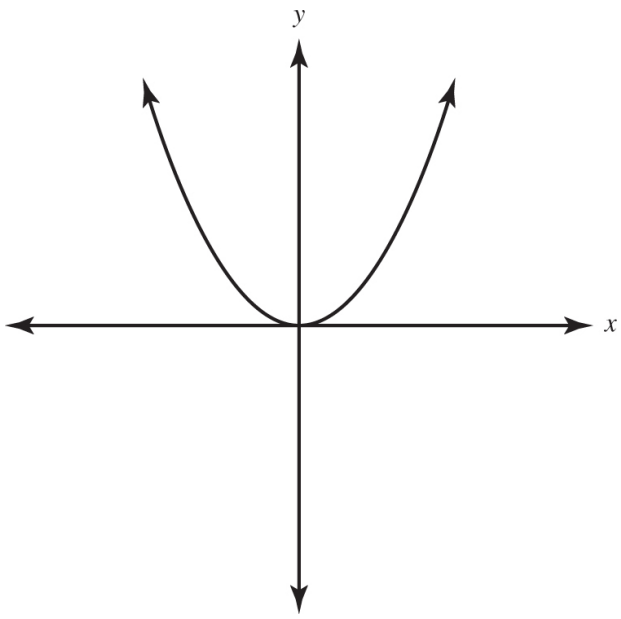


Figure 2.13

For $g(x) = x^3$, the leading coefficient (1) is positive, and the degree (3) is odd. In Figure 2.14, it can be verified that the graph falls on the left, $\lim_{x \to -\infty} g(x) = -\infty$, and rises on the right, $\lim_{x \to \infty} g(x) = +\infty$.

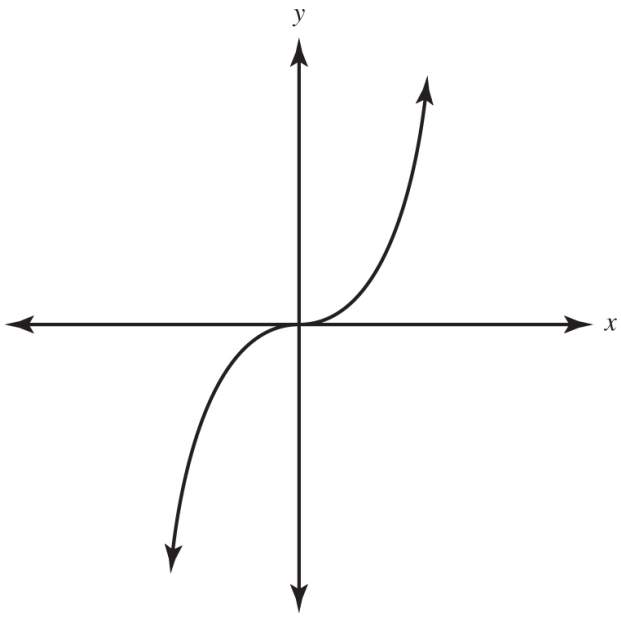


Figure 2.14

Example

State the left and right end behavior of $h(x) = -4x^3 - 17x^2 - 14x + 6$ using limit notation. Confirm your answer graphically.

Solution

The leading coefficient of $h(x)$ is -4, which is negative, and the degree of $h(x)$ is 3, which is odd. Therefore, $\lim_{x \to -\infty} h(x) = +\infty$ and $\lim_{x \to \infty} h(x) = -\infty$. This is confirmed by the graph in Figure 2.15, which rises on the left and falls on the right.

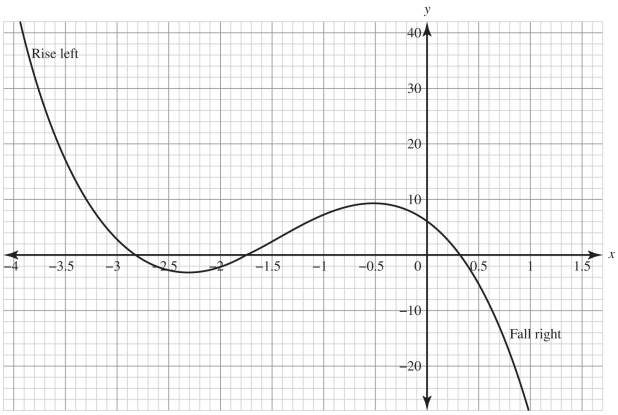


Figure 2.15

2.4 Graphs of Polynomial Functions

Using information from all the previous sections in this chapter helps to graph polynomial functions without the use of a graphing calculator. Below is a summary of a seven-step strategy for graphing a polynomial function.

Seven-Step Strategy for Graphing a Polynomial Function

1. State the degree of the function: this will help determine how many *x*-intercepts the graph may have.
2. State the number of possible relative extrema and points of inflection: this will help determine how the graph will bend and if there will be any relative minima or maxima.
3. Test for end behavior: examine the leading coefficient and degree to determine the end behavior of the graph.
4. Find the *x*-intercept(s): completely factor the polynomial and find its zeros.
5. State the multiplicity of each zero: this will help determine the behavior of the graph at each *x*-intercept.
6. Find the y-intercept: evaluate the polynomial function at $x = 0$.

7. Sketch the graph using the above steps.

› Example

Sketch the graph of the polynomial function $f(x) = 2x^3 + 2x^2 - 8x - 8$.

✓ Solution

Use the seven-step strategy.

STEP 1 State the degree

Degree = 3, the function will have at most 3 real zeros so at most 3 x-intercepts

STEP 2 State any relative minima/maxima and points of inflection

3 − 1 = 2, there will be at most 2 relative minima/maxima

3 − 2 = 1, there will be at most 1 point of inflection

STEP 3 Test for end behavior

Degree = 3, odd

Leading coefficient = 2, positive

$\lim_{x \to -\infty} f(x) = -\infty$: graph will fall on the left

$\lim_{x \to \infty} f(x) = +\infty$: graph will rise on the right

STEP 4 Find the x-intercept(s)

Set the equation equal to 0 and factor completely. Then solve for x.

$$\begin{aligned} 2x^3 + 2x^2 - 8x - 8 &= 0 \\ 2(x^3 + x^2 - 4x - 4) &= 0 \\ 2[x^2(x + 1) - 4(x + 1)] &= 0 \\ 2(x^2 - 4)(x + 1) &= 0 \\ 2(x - 2)(x + 2)(x + 1) &= 0 \end{aligned}$$

Divide by 2 on both sides of the equation:

$$(x-2)(x+2)(x+1)=0$$

The zeros are $x = -2, 2, -1$.

The x-intercepts are $(-2, 0), (2, 0), (-1, 0)$.

STEP 5 State the multiplicity of each zero

Using Step 4, $f(x)$ is written as a product of linear factors. Each factor has a power of 1, so each zero has a multiplicity of 1. Therefore, the graph of $f(x)$ crosses through the x-axis at each x-intercept.

STEP 6 Find the y-intercept

$$f(0)=2(0)^3+2(0)^2-8(0)-8=-8$$

The y-intercept is at the point $(0, -8)$.

STEP 7 Graph

First graph the information found in Steps 1−6 on a set of axes as shown in Figure 2.16.

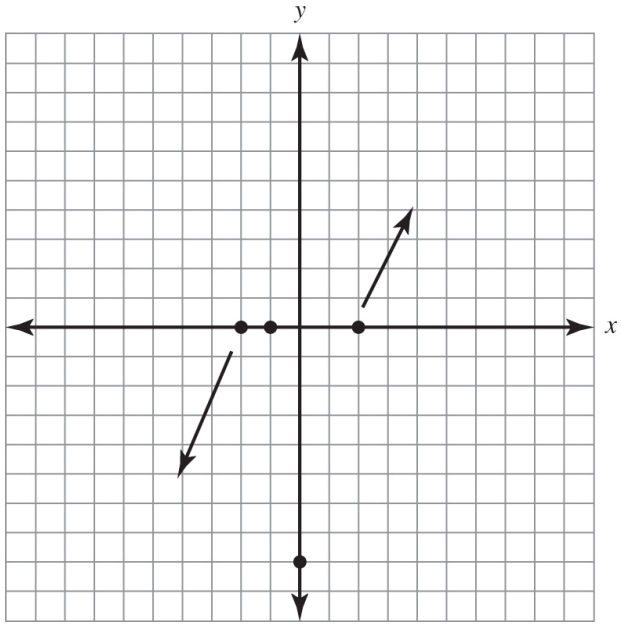


Figure 2.16

Then connect with smooth, continuous curves as shown in Figure 2.17. Check to see that there are at most 2 relative extrema and 1 point of inflection.

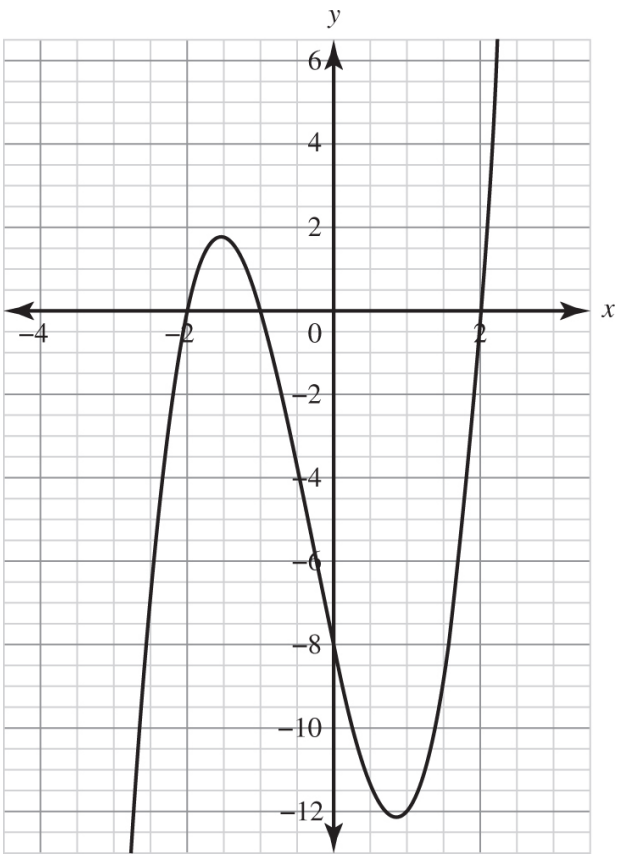


Figure 2.17

Note: Only the x-axis is scaled since the location of the relative extrema and point of inflection are being estimated. Since it is a sketch, the primary concern is the overall shape of the graph. The exact location of relative extrema and points of inflection is saved for a calculus course.

Example

Sketch the graph of the polynomial function $g(x) = -3(x-1)^2(x+1)(x-5)$.

✓ Solution

Utilize the seven-step strategy.

STEP 1 State the degree

The function $g(x)$ is already written in terms of its linear factors. To find the degree, find the sum of the multiplicities.

Degree = 2 + 1 + 1 = 4, the function will have at most 4 real zeros so at most 4 x-intercepts

STEP 2 State any relative minima/maxima and points of inflection

4 − 1 = 3, there will be at most 3 relative minima/maxima

4 − 2 = 2, there will be at most 2 points of inflection

STEP 3 Test for end behavior

Degree = 4, even

To find the leading coefficient, multiply all the leading coefficients in each term together.

Leading coefficient = $(-3)(1)^2(1)(1) = -3$, negative

$\lim_{x \to -\infty} g(x) = -\infty$: graph will fall on the left

$\lim_{x \to \infty} g(x) = -\infty$: graph will fall on the right

STEP 4 Find the x-intercept(s)

The function $g(x)$ is already written as a product of linear factors. Setting each factor equal to 0 shows that the zeros are $x = 1$, $x = -1$, $x = 5$. Therefore, the x-intercepts are (1, 0), (−1, 0), and (5, 0).

STEP 5 State the multiplicity of each zero

Table 2.8 shows a way to organize the information.

Table 2.8

Factor	$(x-1)^2$	$(x+1)$	$(x-5)$
Zero	$x = 1$	$x = -1$	$x = 5$

Multiplicity	2	1	1
Graph Behavior	Tangent to x-axis	Crosses x-axis	Crosses x-axis

STEP 6 Find the y-intercept

$$g(0) = -3(0 - 1)^2(0 + 1)(0 - 5) = -3(-1)^2(1)(-5) = 15$$

The y-intercept is at the point (0, 15).

STEP 7 Graph

First graph the information found in Steps 1−6 on a set of axes as shown in Figure 2.18.

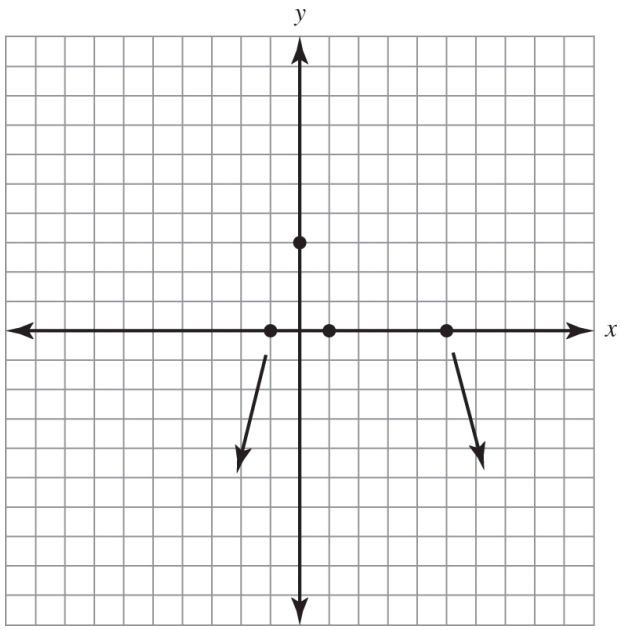


Figure 2.18

Then connect with smooth, continuous curves as shown in Figure 2.19. Check to see that there are at most 3 relative extrema and 2 points of inflection.

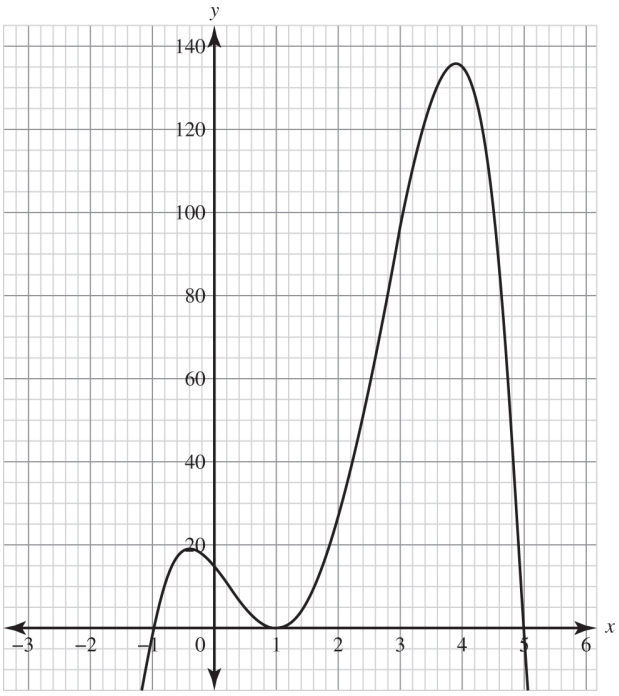


Figure 2.19

Note: Only the *x*-axis is scaled since the location of the relative extrema and points of inflection are being estimated. Since it is a sketch, the primary concern is the overall shape of the graph. The exact location of relative extrema and points of inflection is saved for a calculus course.

Example

Write a possible polynomial function for the graph shown in Figure 2.20.

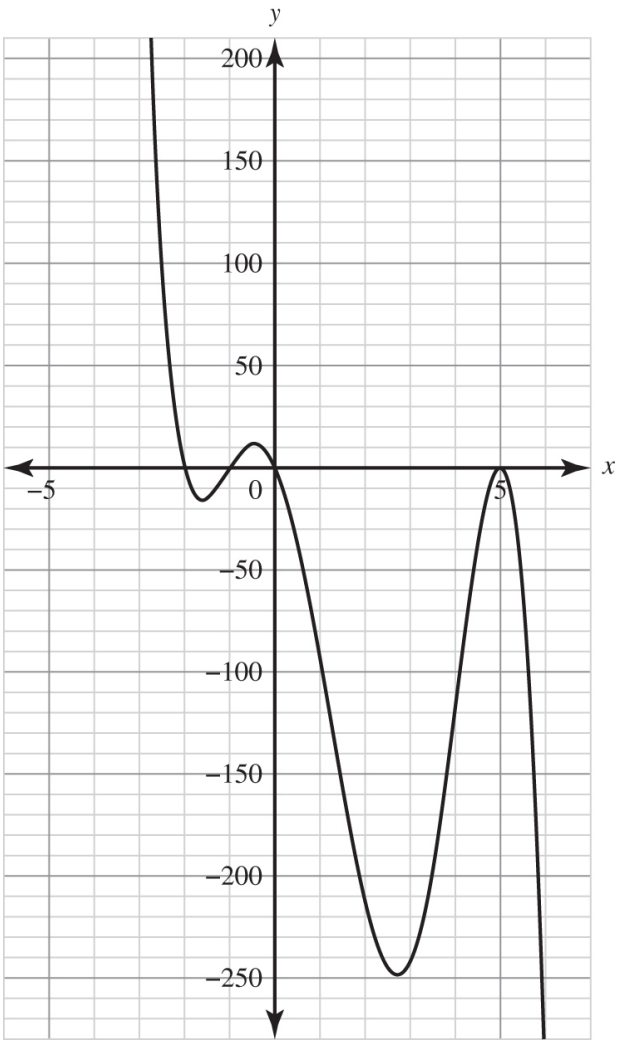


Figure 2.20

✓ Solution

Utilizing the seven-step strategy for graphing helps to identify important parts of the graph to pay attention to when writing an equation for a possible polynomial function.

STEP 1 Find a possible degree of the function

Since there are 4 x-intercepts, the degree is at least 4.

STEP 2 State the number of possible relative extrema and points of inflection

There are two relative minima and two relative maxima, meaning $n - 1 = 4$,

determining that the degree is at least 5.

From left to right, the graph is concave up, then concave down, then concave up, and then concave down again. Therefore, there are 3 points of inflection. This means $n - 2 = 3$, proving the degree is 5.

STEP 3 Test for end behavior

The graph rises on the left and falls on the right. Therefore, the leading coefficient is negative, and the degree is odd.

STEP 4 Find the x-intercept(s)

The x-intercepts are $(-2, 0)$, $(-1, 0)$, $(0, 0)$, and $(5, 0)$.

STEP 5 State the multiplicity of each zero

Table 2.9 organizes all of the information.

Table 2.9

x-intercept	$(-2, 0)$	$(-1, 0)$	$(0, 0)$	$(5, 0)$
Zero	$x = -2$	$x = -1$	$x = 0$	$x = 5$
Factor	$(x + 2)$	$(x + 1)$	$(x - 0) = x$	$(x - 5)$
Graph Behavior	Crosses x-axis	Crosses x-axis	Crosses x-axis	Tangent to x-axis
Multiplicity	Odd	Odd	Odd	Even

STEP 6 Find the y-intercept

The y-intercept is at $(0, 0)$.

STEP 7 Write a possible function for the graph of the polynomial

$$f(x) = -x(x + 2)(x + 1)(x - 5)^2$$

Note: There are infinitely many possible solutions. You must check that the sum of the multiplicities is at least 5, the leading coefficient is negative, and the correct multiplicities are given for each factor.

2.5 Equivalent Representations of Polynomial Expressions

A binomial such as $(a + b)^n$ may be expanded by repeated multiplication using $(a + b)$ as a factor n times or by using an algebraic formula called the binomial theorem.

The binomial theorem shows how to expand a binomial to a given power without performing repeated multiplication.

The Binomial Theorem

If n is a positive whole number, then

$(a + b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + \ldots + {}_nC_ra^0b^n$

where ${}_nC_r = \frac{n!}{r!(n-r)!}$.

When the values of ${}_nC_r$ are arranged in a triangular pattern in which the rows correspond to successive values of n, starting with $n = 0$, the result is an array of numbers called Pascal's triangle, shown in Figure 2.21 below.

row 0:	${}_0C_0$	1
row 1:	${}_1C_0$ ${}_1C_1$	1 1
row 2:	${}_2C_0$ ${}_2C_1$ ${}_2C_2$	1 2 1
row 3:	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$	1 3 3 1
row 4:	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$	1 4 6 4 1
row 5:	${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$	1 5 10 10 5 1
⋮	…	…

Figure 2.21

In Pascal's triangle, each number after 1 is the sum of the two numbers directly above it. Pascal's triangle makes it possible to find the value of ${}_nC_r$ for different values of n and r, where n is the row number and $r = 0$ corresponds to the first entry in each row.

Example

Write the expansion of $(x - 3y)^4$.

Solution

Use the binomial theorem to expand $(x - 3y)^4$, where $a = x$, $b = -3y$, and $n = 4$.

$$(x-3y)^4 = {}_4C_0x^4(-3y)^0 + {}_4C_1x^3(-3y)^1 + {}_4C_2x^2(-3y)^2 + {}_4C_3x^1(-3y)^3 + {}_4C_4x^0(-3y)^4$$

Evaluate the coefficients in front of each term either by using the formula ${}_nC_r = \frac{n!}{r!(n-r)!}$ or by copying the values from row 4 of Pascal's triangle in Figure 2.21.

$$\begin{aligned}(x-3y)^4 &= 1x^4 + 4x^3(-3y) + 6x^2(-3y)^2 + 4x(-3y)^3 + 1(-3y)^4 \\ &= x^4 - 12x^3y + 54x^2y^2 - 108xy^3 + 81y^4\end{aligned}$$

Example

Write the expansion of $(x + y)^6$.

Solution

Use the binomial theorem to expand $(x + y)^6$, where $a = x$, $b = y$, and $n = 6$.

$$(x+y)^6 = {}_6C_0x^6y^0 + {}_6C_1x^5y^1 + {}_6C_2x^4y^2 + {}_6C_3x^3y^3 + {}_6C_4x^2y^4 + {}_6C_5x^1y^5 + {}_6C_6x^0y^6$$

Evaluate the coefficients in front of each term either by using the formula ${}_nC_r = \frac{n!}{r!(n-r)!}$ or by using the values from row 6 of Pascal's triangle. Figure 2.22 shows how to obtain row 6 from row 5 of Pascal's triangle. Write 1 as

the first and the last member of row 6. Then obtain each of the remaining numbers in row 6 by adding the two numbers directly above from row 5.

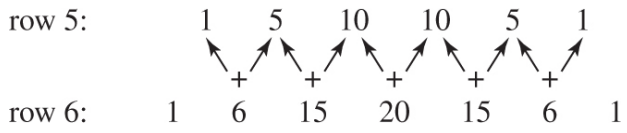


Figure 2.22

$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

2.6 Polynomial Inequalities

The key to solving polynomial inequalities comes from the geometric property that the graph of $y = f(x)$ lies above the x-axis when $f(x) > 0$ and lies below the x-axis when $f(x) < 0$.

The solutions of $f(x) > 0$ are the numbers x for which the graph of $f(x)$ lies above the x-axis. The solutions of $f(x) < 0$ are the numbers x for which the graph of $f(x)$ lies below the x-axis.

Solving Polynomial Inequalities

1. Write the inequality in one of these forms:

   $f(x) > 0$ $f(x) \geq 0$ $f(x) < 0$ $f(x) \leq 0$

2. Determine the roots of $f(x)$.
3. Use test points to determine whether the graph of $f(x)$ is above or below the x-axis on each of the intervals determined by the roots.
4. Find the solutions of the inequality.

Example

Solve: $2x^3 - 15x < x^2$

✓ Solution

1. Write the inequality in the form $f(x) < 0$:

$$2x^3 - x^2 - 15x < 0$$

2. Determine the roots of $f(x) = 2x^3 - x^2 - 15x$.
   Factor the function: $2x^3 - x^2 - 15x = x(2x^2 - x - 15) = x(2x + 5)(x - 3)$
   The roots are $x = 0$, $x = -\frac{5}{2}$, and $x = 3$.
3. Plot the roots on a number line, as shown in Figure 2.23. Determine the sign of the polynomial on each of the intervals determined by the roots. Each root has an open circle since the inequality is strictly less than 0.

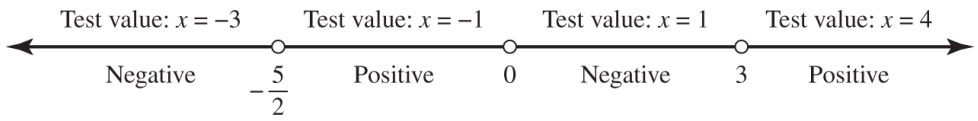


Figure 2.23

4. Since $f(x) < 0$, the solution sets lie in the intervals where $f(x)$ is negative. The solution intervals are $\left(-\infty, -\frac{5}{2}\right) \cup (0, 3)$.

› Example

Solve: $(x + 15)(x - 2)^6(x - 10) \leq 0$

✓ Solution

1. The inequality is already in the form $f(x) \leq 0$.
2. Determine the roots of $f(x) = (x + 15)(x - 2)^6(x - 10)$.
   The function is already factored.
   The roots are $x = -15$, $x = 2$, and $x = 10$.

3. Plot the roots on a number line, as shown in Figure 2.24. Determine the sign of the polynomial on each of the intervals determined by the roots. Each root has a closed circle since the inequality is less than or equal to 0.

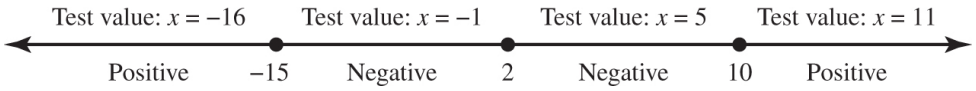


Figure 2.24

4. Since $f(x) \leq 0$, the solution sets lie in the intervals where $f(x)$ is negative. The solution intervals are $[-15, 2] \cup [2, 10]$. The reason why the interval is broken up at $x = 2$ is because at $x = 2$, the function equals 0, which is not negative.

Multiple-Choice Questions

1. Three of the roots of a polynomial equation with rational coefficients are -3, $2i$, and $-4 + 5\sqrt{6}$. What is the lowest possible degree of the equation?

   (A) 3
   (B) 4
   (C) 5
   (D) 6

2. Let $f(x)$ represent a third-degree polynomial function with real coefficients. When $y = f(x)$ is graphed, it is found that $f(3) = -1$ and $f(4) = 2$. Which statement *must* be true?

   (A) The function $f(x)$ has exactly one real zero in the interval $[3, 4]$.
   (B) The function $f(x)$ has exactly one real zero in the interval $[-1, 2]$.
   (C) The function $f(x)$ has exactly one or three real zeros in the interval $[3, 4]$.

(D) The function $f(x)$ has exactly one or two real zeros in the interval $[-1, 2]$.

3. Let $c(x)$ represent a fourth-degree polynomial function with real coefficients. When $y = c(x)$ is graphed, it is found that $c(5) = 3$ and $c(9) = 4$. Which statement *must* be true?

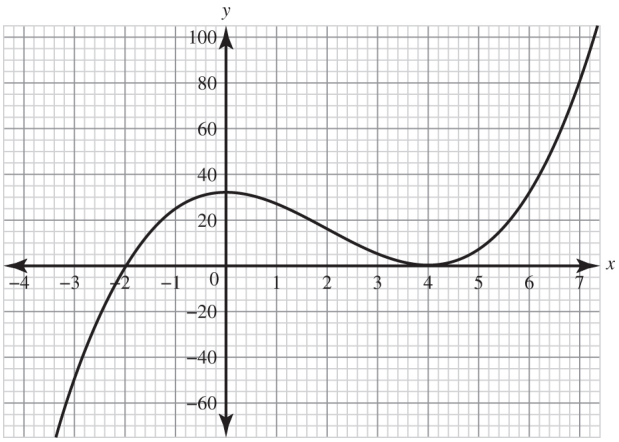
(A) The function $c(x)$ has an odd number of real zeros in the interval $[5, 9]$.

(B) The function $c(x)$ has either no real zeros or an even number of real zeros in the interval $[5, 9]$.

(C) The function $c(x)$ has at least one real zero in the interval $[3, 4]$.

(D) No conclusion is possible.

4. Based on the accompanying graph of the polynomial function $y = h(x)$, which statement *must* be true?



(A) The degree of the function h is even.

(B) The leading coefficient of the function h is negative.

(C) $x = -2$ is a zero of multiplicity 2, and $x = 4$ is a zero of multiplicity 1.

(D) Two of the factors of the function h are $(x + 2)$ and $(x - 4)$.

5. Given the nonconstant polynomial function $f(x)$ whose domain is $[a, e]$ where $a < b < c < d < e$, if $f(b) = f(c) = f(d) = 0$ and no other values of x in the domain are such that $f(x) = 0$, which statement *must* be true?

(A) There is only one relative extremum in the interval $[a, e]$.

(B) There are two relative extrema in the interval $[a, e]$.

(C) $f(a) > 0$ and $f(e) > 0$.

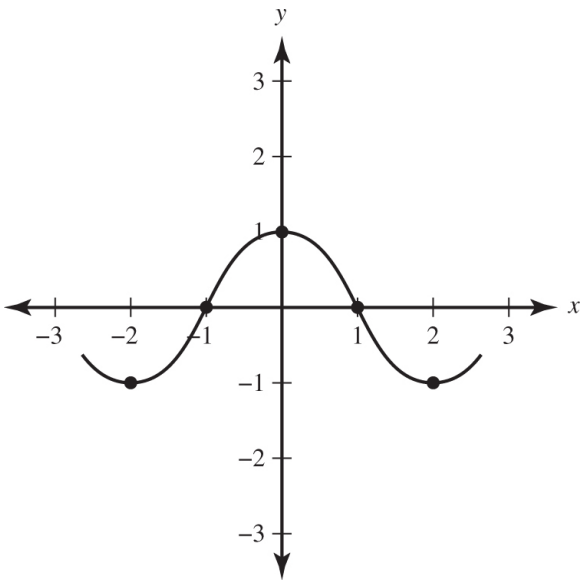
(D) The graph falls on the left and right.

6. Let f be a polynomial function that is even and continuous on the closed interval $[-3, 3]$. The function f has the properties indicated in the table below.

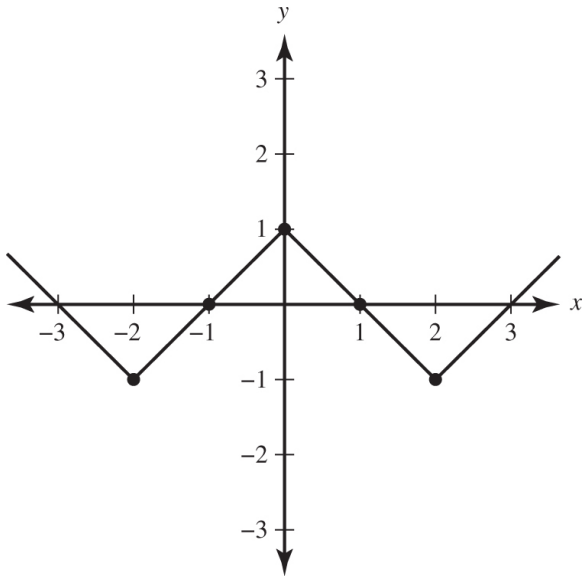
x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	1	Positive	0	Negative	-1	Negative

Which of the following could be a graph of f ?

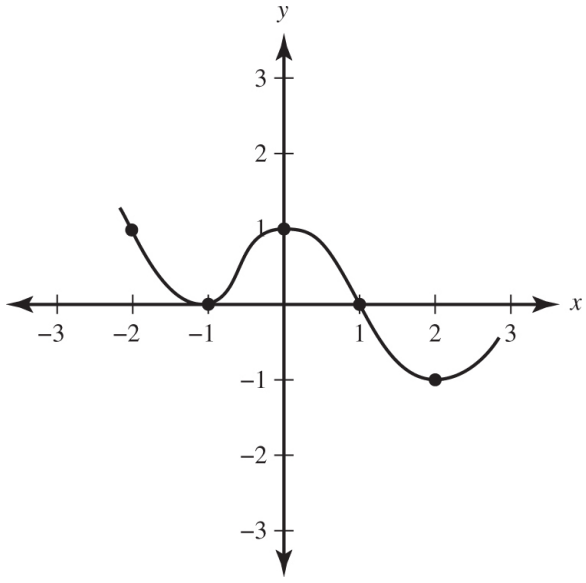
(A)



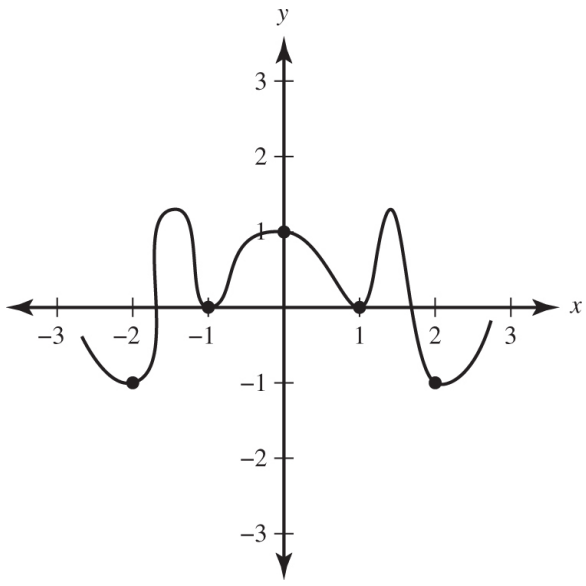
(B)



(C)



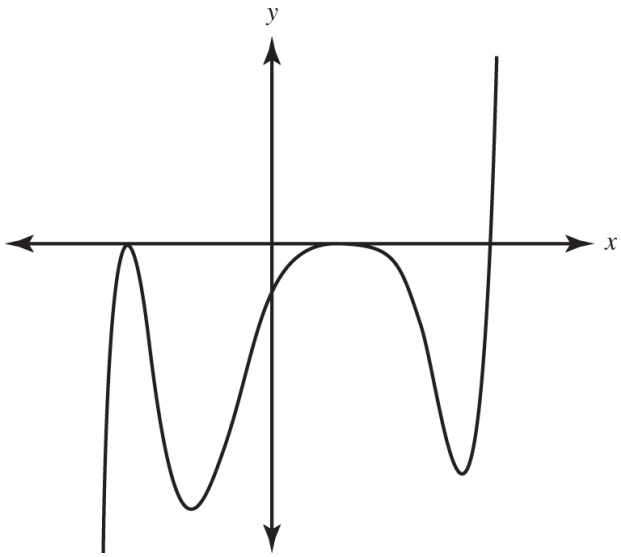
(D)



7. For what value of k is $(x + 1)$ a factor of the function $g(x) = x^4 + 6x^3 + k x^2 - 26x - 24$?

(A) -1
(B) 1
(C) 3
(D) 43

8. Which of the following is a possible polynomial function for the graph shown below?



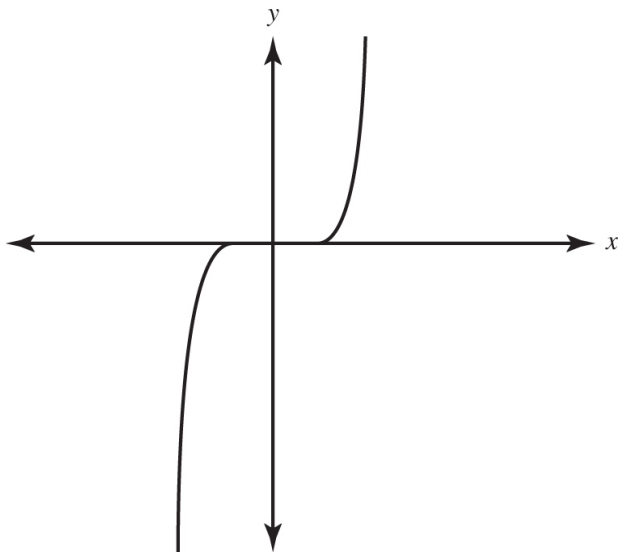
(A) $f(x) = (x - 5)^2 (x + 1)^4(x + 3)$

(B) $h(x) = (x - 5)(x + 1)^3 (x + 3)^2$

(C) $g(x) = (x + 5)^2 (x - 1)^4(x - 3)$

(D) $m(x) = (x + 5)(x - 1)^3 (x - 3)^2$

9. Which of the following statements must be true about the function that is represented by the graph shown below?



(A) $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$

(B) $f(-x) = f(x)$

(C) $f(-x) = -f(x)$

(D) This function has no real zeros.

10. Let g be a polynomial function over the interval $[a, e]$ with the properties listed in the table below.

x	(a, b)	(b, c)	(c, d)	(d, e)	(e, f)
Average Rate of Change	Increasing	Increasing	Decreasing	Increasing	Decreasing

Which of the following statements *must* be true?

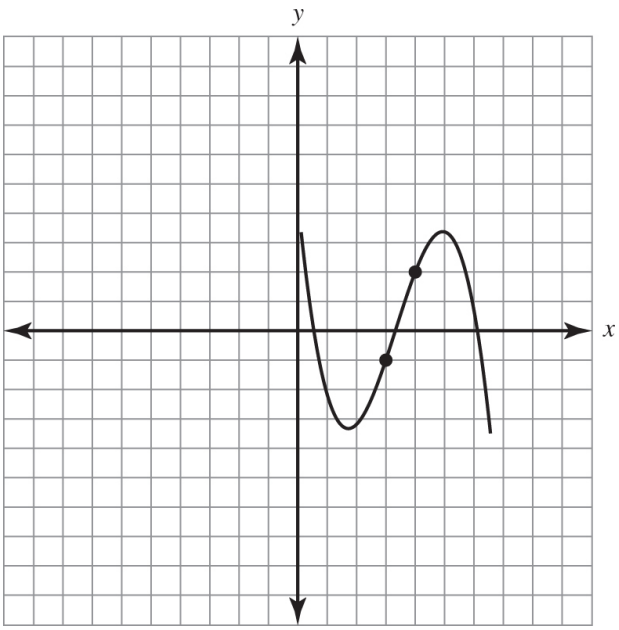
(A) There are two relative extrema.
(B) There are three relative extrema.
(C) There are two points of inflection.
(D) There are three points of inflection.

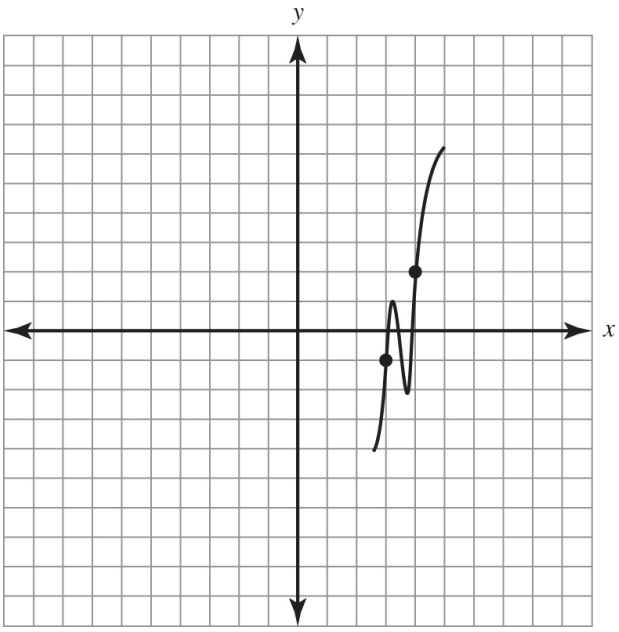
Answer Explanations

1. **(C)** Imaginary and irrational roots come in complex conjugate pairs. So the roots are really -3, $2i$, $-2i$, $-4+5\sqrt{6}$, and $-4-5\sqrt{6}$. If each of these roots occurs only once, the lowest possible degree of the equation would be 5.

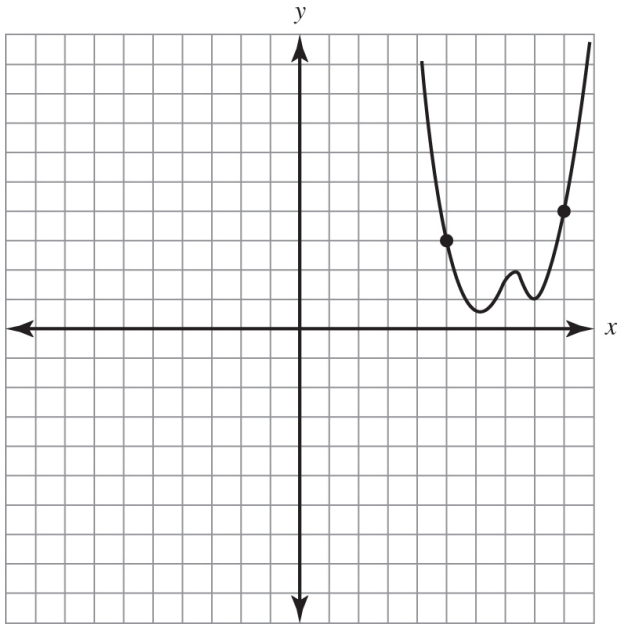
2. **(C)** Since $f(x)$ is a third-degree polynomial, the function will have at most three zeros and the graph will have at most three x-intercepts. Since the output values change from negative to positive over the interval [3, 4], the graph must cross the x-axis at least once. Since no other function values were given, you cannot assume that the graph crosses the x-axis only once. Below are two possible scenarios that would satisfy the given information.

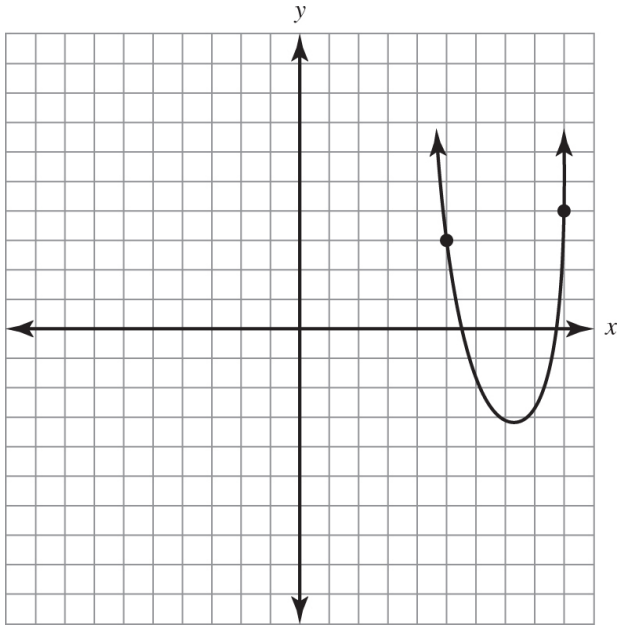
   These graphs show how it is possible to have one or three zeros in the interval.

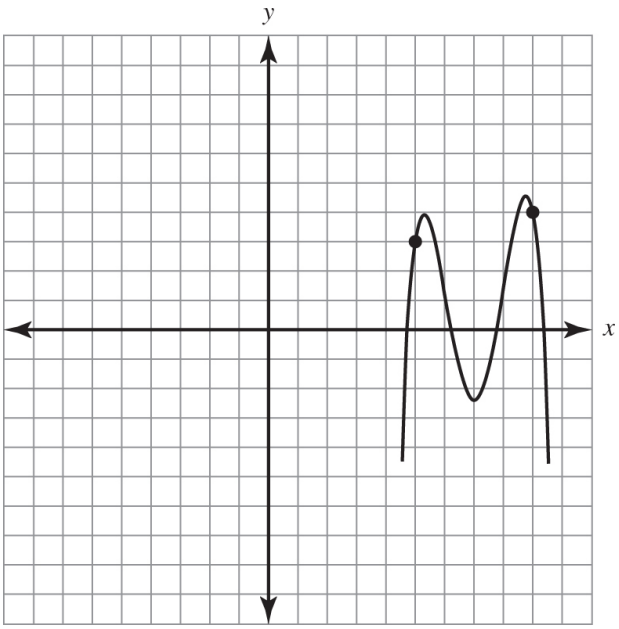




3. **(B)** It is given that $c(x)$ is a fourth-degree polynomial. This means that it can have at most four real zeros. This also means that its end behavior either rises on both the left and right or falls on both the left and right. What that guarantees is that if this function does cross the x-axis, it must cross again in order to have the correct end behavior. Therefore, if this function does have real zeros, they will occur an even number of times. Below are possible scenarios for the given behavior.


y
x


y
x



The first graph on the previous page has no real zeros, the second graph on the previous page has two real zeros, and the graph above has four real zeros. It is possible to have a graph that is tangent to the x-axis to fit the given information, but that would mean that zero has an even multiplicity and would count as two zeros.

4. (D) Since the end behavior of the graph falls on the left and rises on the right, the function must have an odd degree and a positive leading coefficient. Since the graph crosses the x-axis at $x = -2$, its factor $(x + 2)$ must have an odd multiplicity. Since the graph is tangent to the x-axis at $x = 4$, its factor $(x - 4)$ must have an even multiplicity.

5. (B) Between every two distinct real zeros of a nonconstant polynomial function there must be at least one input value corresponding to a local maximum or local minimum. Since it is given that there are exactly three and only three zeros for this function, b, c, and d, there exists a zero between $x = b$ and $x = c$ and a second zero between $x = c$ and $x = d$.

6. (A) Since f is a polynomial function, there must be smooth bends in the graph, which eliminates Choice (B) with its sharp corners. It was given that f is an even function. So it must be symmetric with respect to the y-axis, which eliminates Choice (C). Choice (D) is eliminated because it

3

Rational Functions

Learning Objectives

In this chapter, you will learn:

- ➜ The different properties of rational functions
- ➜ How to graph rational functions
- ➜ How to divide rational functions
- ➜ How to solve rational inequalities

A rational function is represented as a quotient of two polynomial functions such as $h(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions. Unlike polynomial functions, graphs of rational functions may be discontinuous, meaning that there are breaks in the graph. These discontinuities may be asymptotes or holes on the graph.

Since a rational function gives a measure of the relative size of the polynomial function in the numerator compared with the polynomial function in the denominator for each value in the domain, the end behavior of a rational function is found differently from that of a polynomial function. When discussing the behavior of a rational function, it is helpful to know any intercepts, asymptotes, holes, and end behavior of the graph.

3.1 Equivalent Representations of Rational Expressions: Long Division

As with polynomial functions, the factored form of a rational function provides information about zeros, x-intercepts, asymptotes, holes, domain, and range. However, the standard form of rational functions, just like with polynomial functions, is also useful in that it can reveal information about the end behavior of the function.

With polynomial functions, factoring was an important technique when rewriting in equivalent forms. For rational functions, using polynomial long division is necessary. Polynomial long division is an algebraic process that, like numerical long division, involves a quotient and remainder. If $h(x) = \frac{f(x)}{g(x)}$, then $f(x)$ can be rewritten as $f(x) = g(x)q(x) + r(x)$, where q is the quotient, r is the remainder, and the degree of r is less than the degree of g.

The long division of polynomials in standard form follows a pattern like that for the long division of whole numbers.

Being able to represent polynomial and rational functions in different, equivalent forms is an important skill as they each help answer questions about the function.

Example

Find the quotient and remainder when x^3 is divided by $x^2 - 9$ using long division.

Solution

Table 3.1 shows the steps to solve this problem.

Table 3.1

1. Divide the first term of the dividend, x^3, by the first term of the divisor, x^2.	$x^2 - 9\overline{)\,x^3}$

$\frac{x^3}{x^2} = x$ Write x above the leading term of the dividend.	
2. Multiply x by the divisor $x^2 - 9$. $x(x^2 - 9) = x^3 - 9x$ Write the result on the line below the dividend, aligning like terms.	x $x^2 - 9 \overline{)x^3}$ $x^3 - 9x$
3. Subtract and bring down the remaining terms.	x $x^2 - 9 \overline{)x^3}$ $-(x^3 - 9x)$ $9x$

The process is finished here because dividing $9x$ by $x^2 - 9$ does not result in a monomial. The quotient is x, and the remainder is $9x$.

The solution can be written as:

$$\frac{x^3}{x^2 - 9} = x + \frac{9x}{x^2 - 9}$$

Example

Using long division, write an equivalent form of the rational function $f(x) = \frac{2x^3 + 5x^2 - 3x + 7}{x^2 + 6}$.

Solution

When setting up the long division problem, $x^2 + 6$ is the divisor and $2x^3 + 5x^2 - 3x + 7$ is the dividend. Then follow the steps shown in Table 3.2.

Table 3.2

1. Divide the first term of the dividend, $2x^3$, by the first term of the divisor, x^2.	$2x$ $x^2 + 6 \overline{)2x^3 + 5x^2 - 3x + 7}$

$\frac{2x^3}{x^2} = 2x$ Write $2x$ above the leading term of the dividend.	
2. Multiply $2x$ by the divisor $x^2 + 6$. $2x(x^2 + 6) = 2x^3 + 12x$ Write the result on the line below the dividend, aligning like terms.	$\begin{array}{r} 2x \\ x^2+6\overline{)\,2x^3+5x^2-3x+7} \\ \underline{2x^3\qquad +12x} \end{array}$
3. Subtract and bring down the remaining terms.	$\begin{array}{r} 2x \\ x^2+6\overline{)\,2x^3+5x^2-3x+7} \\ \underline{-(2x^3\qquad +12x)} \\ 5x^2-15x+7 \end{array}$
4. Repeat the first three steps, using $5x^2 - 15x + 7$ as the dividend. $\frac{5x^2}{x^2} = 5$ $5(x^2 + 6) = 5x^2 + 30$	$\begin{array}{r} 2x+5 \\ x^2+6\overline{)\,2x^3+5x^2-3x+7} \\ \underline{-(2x^3\qquad +12x)} \\ 5x^2-15x+7 \\ \underline{-(5x^2\qquad +30)} \\ -15x-23 \end{array}$

The process is finished here because dividing $-15x - 23$ by $x^2 + 6$ does not result in a monomial. The quotient is $2x + 5$, and the remainder is $-15x - 23$.

The solution can be written as:

$$f(x) = \frac{2x^3 + 5x^2 - 3x + 7}{x^2 + 6} = 2x + 5 + \frac{-15x - 23}{x^2 + 6}$$

3.2 Rational Functions and Zeros

The real zeros of a rational function correspond to the real zeros of the numerator for such values in its domain. Graphically, the zeros of the function are also the x-intercepts of the graph of the rational function.

Finding Intercepts

To determine the points at which the graph of the rational function $f(x) = \frac{P(x)}{Q(x)}$ intersects the coordinate axes:

- Set $P(x) = 0$. Any solution of this equation that does not also make $Q(x) = 0$ is an x-intercept.
- Evaluate $f(0)$. The y-intercept is $(0, f(0))$ if it is defined.

› Example

Find the x- and y-intercepts of the rational function $f(x) = \frac{x^2 - 64}{x + 2}$.

✓ Solution

To find the x-intercept(s), set the polynomial function in the numerator equal to 0 and solve for x:

$$\begin{aligned} x^2 - 64 &= 0 \\ (x + 8)(x - 8) &= 0 \\ x &= \pm 8 \end{aligned}$$

Since the denominator does not equal 0 when $x = \pm 8$, the x-intercepts are $(-8, 0)$ and $(8, 0)$.

To find the y-intercept, evaluate $f(0)$:

$$f(0) = \frac{(0)^2 - 64}{(0) + 2} = \frac{-64}{2} = -32$$

The y-intercept is $(0, -32)$.

3.3 Rational Functions and Vertical Asymptotes

An asymptote is a line whose distance from a curve approaches 0 as the curve tends to infinity. The graph of a rational function has a vertical asymptote at a value of x for which the denominator evaluates to 0 but the numerator does not evaluate to 0.

Locating Vertical Asymptotes

Let $f(x) = \frac{P(x)}{Q(x)}$. The line $x = a$ is a vertical asymptote of the graph of function f when $Q(a) = 0$ and $P(a) \neq 0$.

› Example

Find and graph the vertical asymptotes of $f(x) = \frac{x^2}{x^2 - 5x + 6}$.

✓ Solution

To find the vertical asymptotes, set the denominator equal to 0 and solve for x:

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ (x - 3)(x - 2) &= 0 \\ x &= 3,\ x = 2 \end{aligned}$$

Since the numerator does not equal 0 when $x = 3$ or $x = 2$, the vertical asymptotes for f are $x = 3$ and $x = 2$. The vertical asymptotes are graphed with a dashed vertical line as shown in Figure 3.1. For now, the rest of the graph is found by using a graphing calculator.

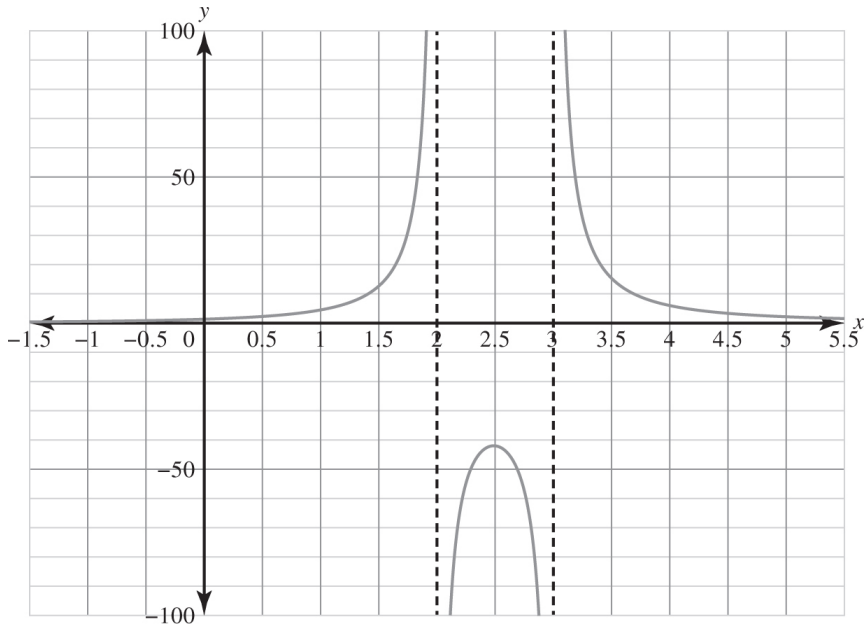


Figure 3.1

Limit notation is also useful here to show the behavior of a rational function at a vertical asymptote. Near a vertical asymptote, $x = a$, of a rational function, the values of the polynomial in the denominator are arbitrarily close to 0, so the values of the rational function increase or decrease without bound. This can be notated by $\lim_{x \to a^-} f(x) = -\infty$ or $\lim_{x \to a^-} f(x) = \infty$. This is read as "the limit as x approaches a from the left of the function $f(x)$ equals negative or positive infinity." "The limit as x approaches a from the left" means the input values begin less than a and then get closer to a.

Notice the graph of the function gets closer and closer to the asymptote without ever intersecting. This is what is meant by the asymptote being a line whose distance from a curve approaches 0 as the curve tends to infinity.

Similarly, coming in from the other direction can be notated by $\lim_{x \to a^+} f(x) = -\infty$ or $\lim_{x \to a^+} f(x) = \infty$. This is read as "the limit as x approaches a from the right of the function $f(x)$ equals negative or positive infinity." "The limit

as x approaches a from the right" means the input values begin greater than a as they get closer to a.

Table 3.3 demonstrates this with a table of values from our previous example using the function $f(x) = \frac{x^2}{x^2 - 5x + 6}$ and its vertical asymptote $x = 2$.

Table 3.3

x	1	1.9	1.999999	2	2.00001	2.01	2.1
$f(x)$	4.5	90	10000000	undefined	−1000000	−1011	−112.2

As x is approaching 2 from the left, the output values get larger and larger, increasing without bound. As x is approaching 2 from the right, the output values are getting smaller and smaller, decreasing without bound.

Therefore, $\lim_{x \to 2^-} f(x) = \infty$ and $\lim_{x \to 2^+} f(x) = -\infty$.

3.4 Rational Functions and Holes

A second type of discontinuity is a removable discontinuity, which is also referred to as a hole because of how it is graphed. A rational function has a removable discontinuity if the multiplicity of a real zero in the numerator is greater than or equal to its multiplicity in the denominator.

Finding the x-Coordinate of a Hole for a Rational Function

Completely factor the polynomials in the numerator and denominator of the rational function. If there is a common factor in the numerator and denominator, it can be removed. The zeros of the common factors represent the x-coordinates of the hole. After finding the coordinates, it can be graphed using an open circle, or hole.

If the graph of a rational function has a hole at $x = c$, then the location of the hole can be determined by examining the output values corresponding to input values arbitrarily close to c. If input values arbitrarily close to c correspond to output values arbitrarily close to L, the hole is located at the point with coordinates (c, L).

Finding the y-Coordinate of a Hole for a Rational Function

After the common factor is removed from the numerator and denominator of the rational function, substitute each x-value of the holes into all the remaining factors of the rational function. The solutions represent the y-coordinate of each respective x-value.

› Example

Find the discontinuities for the function $h(x) = \frac{x^2 + 9x + 8}{x + 8}$ and graph $h(x)$.

✓ Solution

When working with rational functions, factor the polynomials in the numerator and denominator to correctly identify all removable discontinuities:

$$h(x) = \frac{x^2 + 9x + 8}{x + 8} = \frac{(x + 8)(x + 1)}{x + 8}$$

There is a common factor of $(x + 8)$ in the numerator and denominator that can be removed. Note that although this factor is removed, it doesn't disappear and still influences the graph of the function. The zero of the removable factor is the x-coordinate of the hole. The x-coordinate of the hole is $x = -8$.

After removing this discontinuity, the remaining factors make up the rational function. They are used to find the y-coordinate of the hole and contribute to the other behaviors of the rational function:

$$h(x) = x + 1$$

To find the y-coordinate, substitute $x = -8$ into the new function:

$$h(-8) = (-8) + 1 = -7$$

The coordinates of the hole are $(-8, -7)$.

With the removal of this discontinuity, there are no longer any terms left in the denominator, so there are no vertical asymptotes. Using technology, the graph is shown in Figure 3.2.

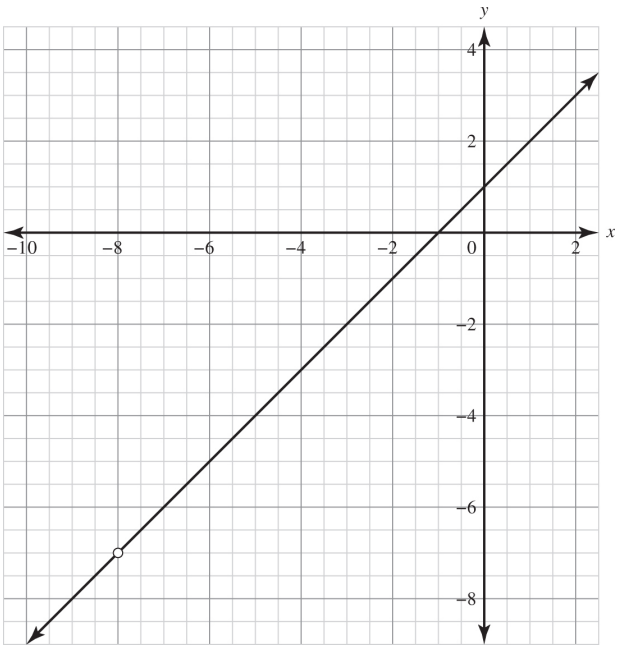


Figure 3.2

It is important to always factor first; then holes can be identified correctly and removed. Any remaining factors in the denominator can be correctly identified as vertical asymptotes.

3.5 Rational Functions and End Behavior

The end behavior of a rational function will be most affected by the polynomial with the greater degree, as its values will dominate the values of the rational function for input values of large magnitude. For input values of large magnitude, a polynomial is dominated by its leading term. Therefore, the end behavior of a rational function can be understood by examining the corresponding quotient of the leading terms.

There are two different possibilities for the end behavior of a rational function, which we call end behavior asymptotes: a horizontal asymptote or a slant asymptote.

Finding End Behavior Asymptotes

Let $f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0}$, where $a_n \neq 0$ and $b_m \neq 0$.

- If $n < m$, the rational function has a horizontal asymptote at $y = 0$.
- If $n = m$, the rational function has a horizontal asymptote at $y = \frac{a_n}{b_m}$.
- If $n > m$, the rational function has no horizontal asymptote. The end behavior asymptote is the quotient polynomial function that is found when the polynomial $P(x)$ is divided by $Q(x)$. If that polynomial is linear, the rational function has a slant asymptote parallel to the graph of the line.

Example

Find the horizontal asymptotes of each of the rational functions. Use technology to graph each.

1. $f(x) = \frac{5x}{x^2 - 9}$
2. $g(x) = \frac{3x^2 - x - 4}{2x^2 + 5}$
3. $h(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1}$

4. $j(x) = \frac{x^3}{x^2 - 9}$

☑ Solution

1. For $f(x)$, the degree of the numerator is 1 and the degree of the denominator is 2. Since $1 < 2$, the rational function has the horizontal asymptote $y = 0$. Figure 3.3 shows the graph of this function.

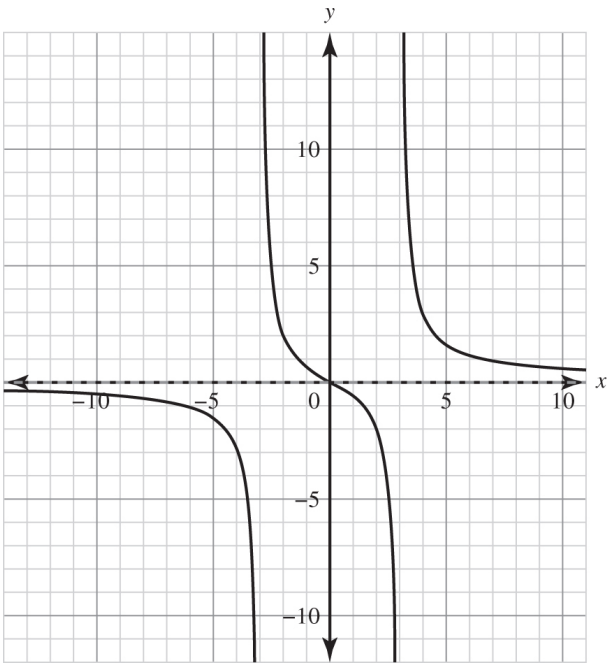


Figure 3.3

As x approaches $\pm \infty$, $f(x)$ approaches 0. This graphically shows how the end behavior of a rational function is shown through its horizontal asymptote. Note that at $x = 0$, the graph intersects the origin, showing that the middle portion of the graph can intersect the horizontal asymptote. The ends of the graph, though, approach the x-axis, or $y = 0$, and follow along the horizontal asymptote, which is graphed as a dashed horizontal line.

On the graph, it appears that there are two vertical asymptotes at $x = -3$ and $x = 3$. This is verified when the denominator of $f(x)$ is factored and set equal to 0.

2. For $g(x)$, the degree of the numerator is 2 and the degree of the denominator is 2. Since 2 = 2, the rational function has the horizontal asymptote $y = \frac{3}{2}$. Figure 3.4 shows the graph of this function.

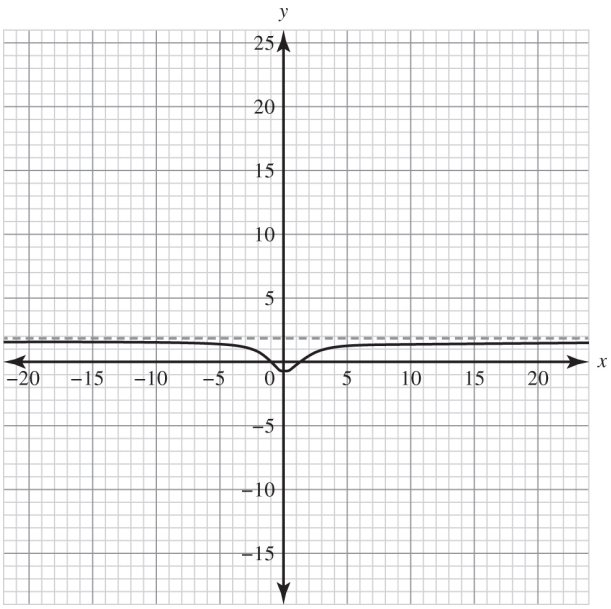


Figure 3.4

3. For $h(x)$, the degree of the numerator is 3 and the degree of the denominator is 1. Since 3 > 1, the rational function has no horizontal asymptote. However, the end behavior asymptote is found by using polynomial long division:

$$h(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1} = x^2 - 2x + 1 + \frac{2}{x - 1}$$

The end behavior asymptote of $h(x)$ is $y = x^2 - 2x + 1$, the quotient after completing long division.

Figure 3.5 shows the graph of $h(x)$. It can also be seen that there is a vertical asymptote at $x = 1$, which is confirmed algebraically by setting the denominator equal to 0.

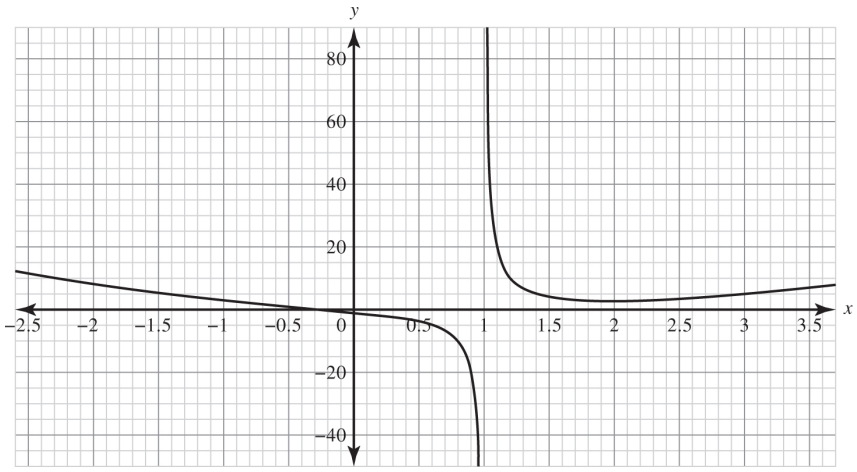


Figure 3.5

Figure 3.6 is another graph of $h(x)$ that shows its end behavior asymptote $y = x^2 - 2x + 1$ as a dashed curve. As x approaches $\pm \infty$, the graph of $h(x)$ and its asymptote appear to be identical.

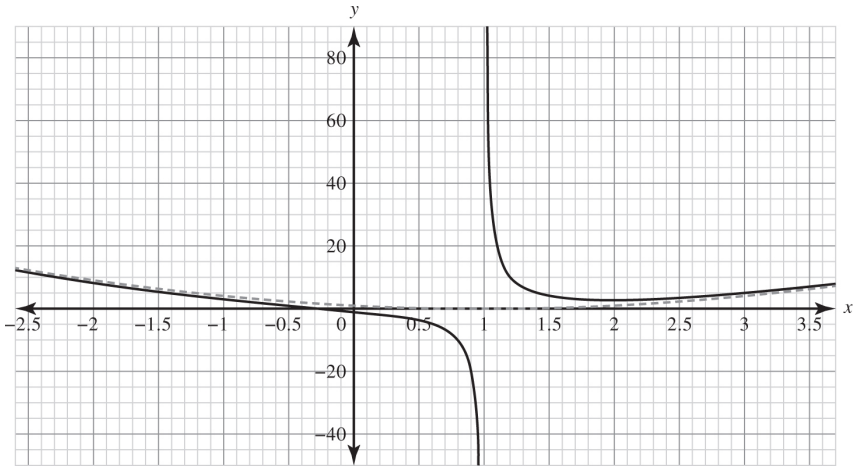


Figure 3.6

4. For $j(x)$, the degree of the numerator is 3 and the degree of the denominator is 2. Since $3 > 2$, the rational function has no horizontal asymptote. However, the end behavior asymptote is found by using polynomial long division:

$$j(x) = \frac{x^3}{x^2 - 9} = x + \frac{9x}{x^2 - 9}$$

The end behavior asymptote of $j(x)$ is $y = x$, the quotient after completing long division. Since this is a linear function, this is known as a slant asymptote.

Figure 3.7 shows the graph of $j(x)$. Vertical asymptotes can be seen at $x = -3$ and $x = 3$, which is confirmed algebraically by setting the denominator equal to 0.

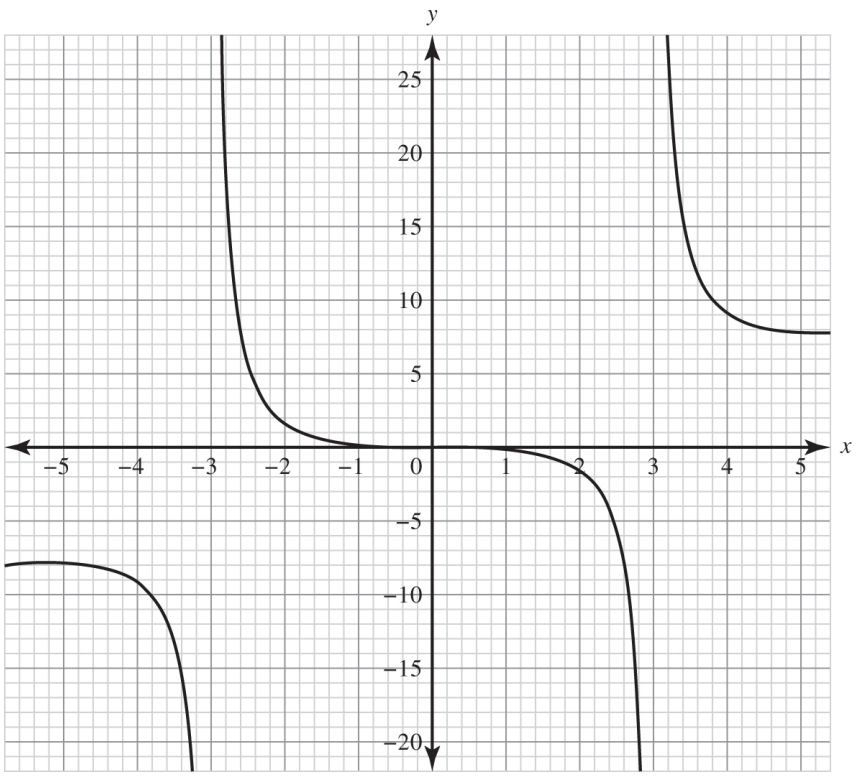


Figure 3.7

Figure 3.8 is another graph of $j(x)$ that shows its end behavior asymptote $y = x$. As x approaches $\pm \infty$, the graph of $j(x)$ and its asymptote appear to be identical as the graph of $j(x)$ runs parallel to the graph of $y = x$.

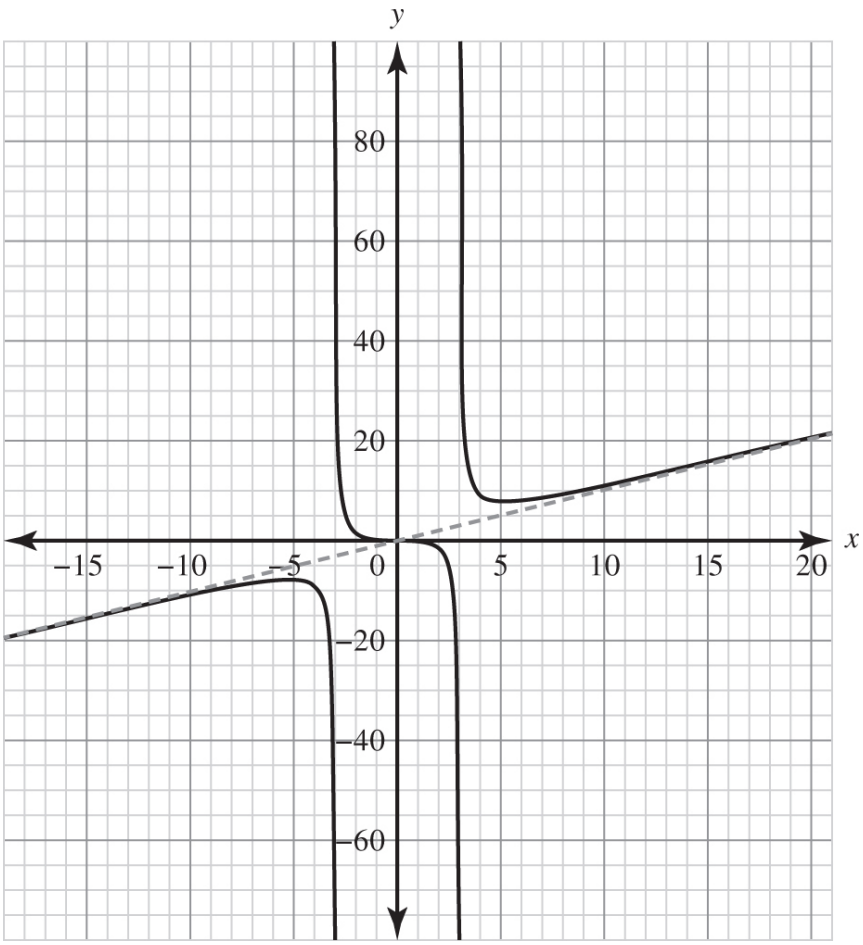


Figure 3.8

It is possible for the graph of a rational function to intersect a horizontal asymptote and a slant asymptote. The graph of a rational function will never intersect a vertical asymptote.

Example

If $f(x) = \dfrac{10x^4 - 2x^2 + 3x + 7}{5x^4 + x^3 - 2}$, evaluate each of the following limits:

(a) $\lim_{x \to -\infty} f(x)$

(b) $\lim_{x \to \infty} f(x)$

✓ Solution

Limits approaching infinity are really asking about the end behavior of a function. Since this is a rational function, the end behavior is found by examining the degrees. The degree of the numerator is 4, and the degree of the denominator is 4. Since the degrees are equal, the graph has a horizontal asymptote at $y = \frac{10}{5} = 2$.

(a) As x gets more negative, the y-values will get closer to 2. Therefore:

$$\lim_{x \to -\infty} f(x) = 2$$

(b) As x gets more positive, the y-values will get closer to 2. Therefore:

$$\lim_{x \to \infty} f(x) = 2$$

These limits can both be verified with the graph of $f(x)$ shown in Figure 3.9.

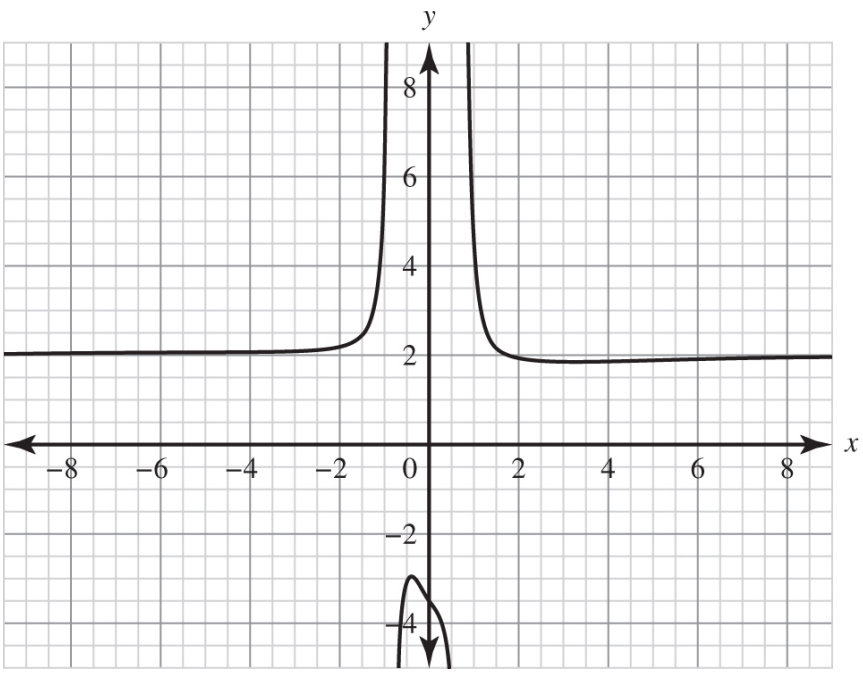


Figure 3.9

When the graph of a rational function, $f(x)$, has a horizontal asymptote $y = b$, where b is a constant, the output values of the rational function get arbitrarily close to b and stay arbitrarily close to b as input values increase or decrease without bound. In other words, $\lim_{x \to -\infty} f(x) = b$ and $\lim_{x \to \infty} f(x) = b$.

3.6 Graphs of Rational Functions

As with polynomial functions, there is a suggested order to follow to graph rational functions.

Seven-Step Strategy for Graphing a Rational Function

1. Factor the numerator and denominator of the rational function to find any holes. Remove any discontinuities.
2. Find all possible x- and y-intercepts.
3. Determine the degree of the numerator and denominator. Determine the end behavior asymptote.
4. Find all vertical asymptotes with any remaining factors in the denominator.
5. Find any symmetry the graph displays by evaluating $f(-x)$.
6. The vertical asymptotes will divide the graph into regions. In each region, graph at least one point that will determine whether the graph will be above or below any horizontal asymptotes.
7. Sketch a graph using the above information.

› Example

Find the key features of the graph of $f(x) = \frac{x^2 - 64}{x^2 - 16}$ and then sketch the graph.

✓ Solution

To find the key features of the graph, utilize the seven-step strategy.

STEP 1 Find any holes

Factor the numerator and denominator, and look for any common factors:

$$f(x) = \frac{x^2 - 64}{x^2 - 16} = \frac{(x - 8)(x + 8)}{(x - 4)(x + 4)}$$

Since there are no common factors, there are no removable discontinuities. The graph of $f(x)$ does not have any holes.

STEP 2 Find all possible x- and y-intercepts

To find the x-intercepts, set the numerator equal to 0 and solve for x:

$$x^2 - 64 = 0, x = \pm 8$$

The x-intercepts are (–8, 0) and (8, 0).

To find the y-intercept, evaluate $f(0)$:

$$f(0) = \frac{(0)^2 - 64}{(0)^2 - 16} = \frac{-64}{-16} = 4$$

The y-intercept is (0, 4).

STEP 3 Find the end behavior asymptote

The degree of the numerator and denominator is 2. Since the degrees are equal, there is a horizontal asymptote. Its equation is $y = \frac{1}{1} = 1$.

STEP 4 Find all vertical asymptotes

Set the denominator equal to 0 and solve for x:

$$(x - 4)(x + 4) = 0, x = \pm 4$$

Since the numerator does not equal 0 at these x-values, the vertical asymptotes are $x = -4$ and $x = 4$.

STEP 5 Evaluate $f(-x)$

$$f(-x) = \frac{(-x)^2 - 64}{(-x)^2 - 16} = \frac{x^2 - 64}{x^2 - 16} = f(x)$$

$f(x)$ is an even function that has symmetry with respect to the y-axis.

STEP 6 Test points

As shown in Table 3.4, choose x-values within each interval broken up by the vertical asymptotes. Then find their corresponding y-values.

Table 3.4

x	-5	1	5
$f(x)$	$\frac{(-5)^2 - 64}{(-5)^2 - 16} = \frac{-39}{9} = -4.333$	$\frac{(1)^2 - 64}{(1)^2 - 16} = \frac{-63}{-15} = 4.2$	$\frac{(5)^2 - 64}{(5)^2 - 16} = \frac{-39}{9} = -4.333$

Since the function is even, the point (1, 4.2) will also lie on the graph.

STEP 7 Graph

Plot and graph everything found in Steps 1–6, as shown in Figure 3.10.

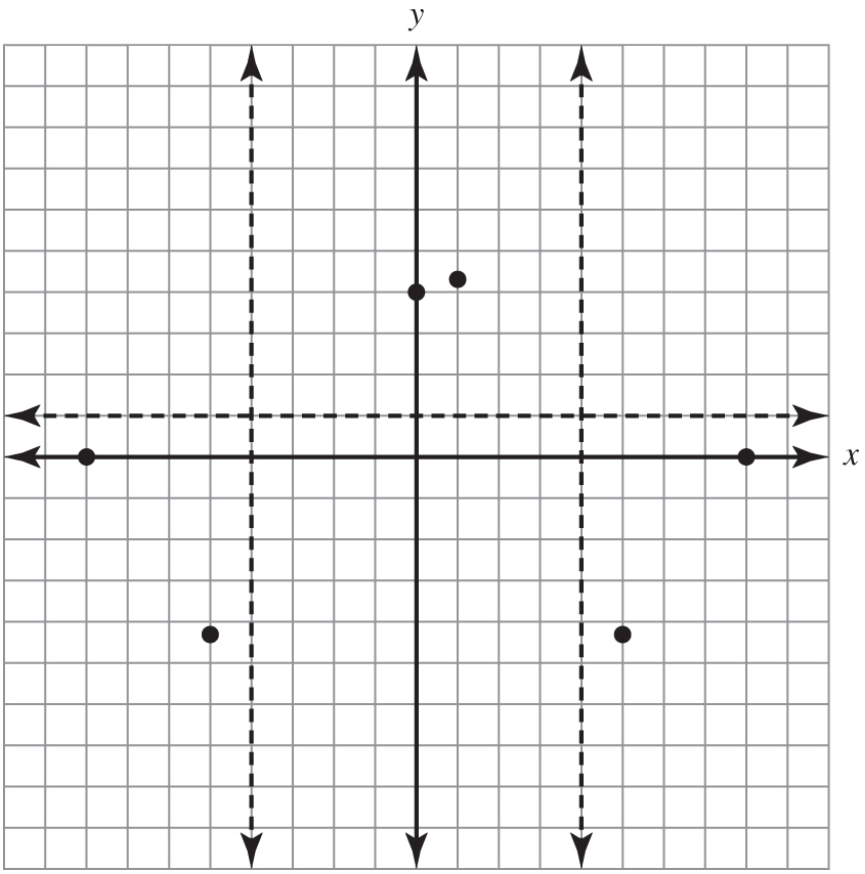


Figure 3.10

To create the graph, as shown in Figure 3.11, follow along the horizontal and vertical asymptotes while passing through the intercepts. Remember to observe the symmetry with respect to the *y*-axis.

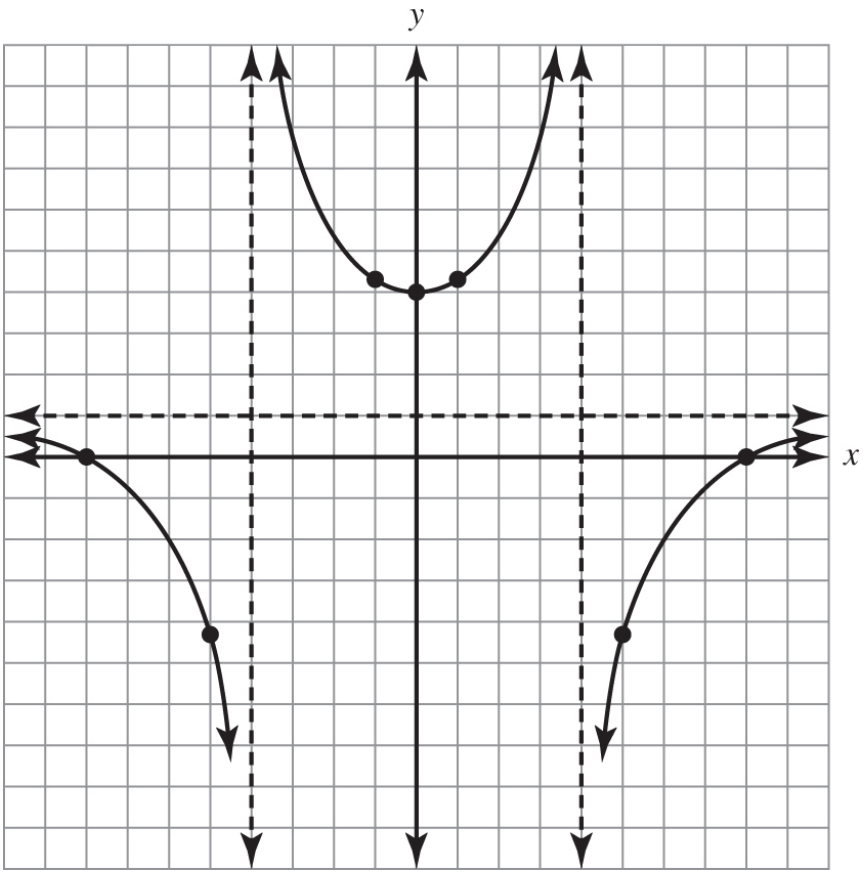


Figure 3.11

Example

Find the key features of the graph of $g(x) = \frac{(x+1)(3x^2-12x+2)}{(x+1)(x-4)}$ and then sketch the graph.

Solution

To find the key features of the graph, utilize the seven-step strategy.

STEP 1 Find any holes

Since the given rational function is already factored, look for any common factors:

$$g(x) = \frac{(x+1)(3x^2-12x+2)}{(x+1)(x-4)}$$

There is a common factor, $(x + 1)$. Remove this discontinuity to produce the following function:

$$g(x) = \frac{3x^2 - 12x + 2}{x - 4}$$

The x-coordinate of the hole is $x = -1$. To find its corresponding y-value, evaluate $g(-1)$:

$$g(-1) = \frac{3(-1)^2 - 12(-1) + 2}{(-1) - 4} = \frac{17}{-5} = -3.4$$

The coordinates of the hole are $(-1, -3.4)$.

The remaining steps will now be determined using the following function: $g(x) = \frac{3x^2 - 12x + 2}{x - 4}$

STEP 2 Find all possible x- and y-intercepts

To find the x-intercepts, set the numerator equal to 0 and solve for x:

$$3x^2 - 12x + 2 = 0$$

To solve for x, use the quadratic formula:

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(2)}}{2(3)} = \frac{12 \pm \sqrt{120}}{6} = \frac{12 \pm \sqrt{4}\sqrt{30}}{6} = \frac{12 \pm 2\sqrt{30}}{6} = \frac{6 \pm \sqrt{30}}{3}$$

The x-intercepts are $\left(\frac{6 - \sqrt{30}}{3}, 0\right) \approx (0.174258, 0)$ and $\left(\frac{6 + \sqrt{30}}{3}, 0\right) \approx (3.825741, 0)$.

To find the y-intercept, evaluate $g(0)$:

$$g(0) = \frac{3(0)^2 - 12(0) + 2}{(0) - 4} = -\frac{2}{4} = -\frac{1}{2}$$

The y-intercept is $\left(0, -\frac{1}{2}\right)$.

STEP 3 Find the end behavior asymptote

The degree of the numerator is 2, and the degree of the denominator is 1. Since $2 > 1$, there is no horizontal asymptote.

Use polynomial long division to find the end behavior asymptote:

$$
\begin{array}{r}
3x \\
x-4\overline{)3x^2-12x+2} \\
\underline{-(3x^2-12x)} \\
2
\end{array}
$$

$$g(x)=\frac{3x^2-12x+2}{x-4}=3x+\frac{2}{x-4}$$

The end behavior asymptote is $y = 3x$. Since it is linear, it is a slant asymptote.

STEP 4 Find all vertical asymptotes

Set the denominator equal to 0 and solve for x:

$$x-4=0, x=4$$

Since the numerator does not equal 0 at this x-value, the vertical asymptote is $x = 4$.

STEP 5 Evaluate $g(-x)$

$$g(-x)=\frac{3(-x)^2-12(-x)+2}{(-x)-4}=\frac{3x^2+12x+2}{-x-4}$$

This does not equal either $g(x)$ or $-g(x)$. This means that $g(x)$ is not an even or an odd function. So there will not be any symmetry with respect to the y-axis or the origin.

STEP 6 Test points

As shown in Table 3.5, choose x-values within each interval broken up by the vertical asymptote. Then find their corresponding y-values.

Table 3.5

x	-2	5
$g(x)$	$\frac{3(-2)^2-12(-2)+2}{(-2)-4}=\frac{38}{-6}=-6.333$	$\frac{3(5)^2-12(5)+2}{(5)-4}=\frac{17}{1}=17$

STEP 7 Graph

Plot and graph everything found in Steps 1–6, as shown in Figure 3.12.

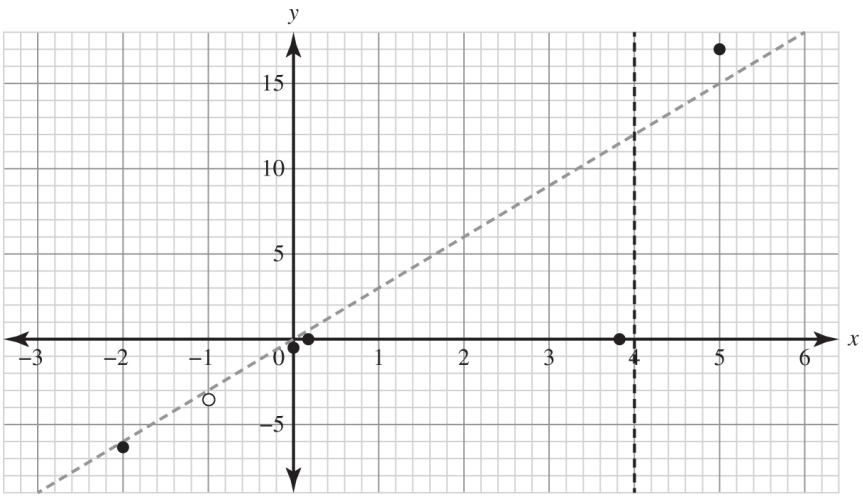


Figure 3.12

To create the graph, as shown in Figure 3.13, follow along the slant and vertical asymptotes while passing through the intercepts.

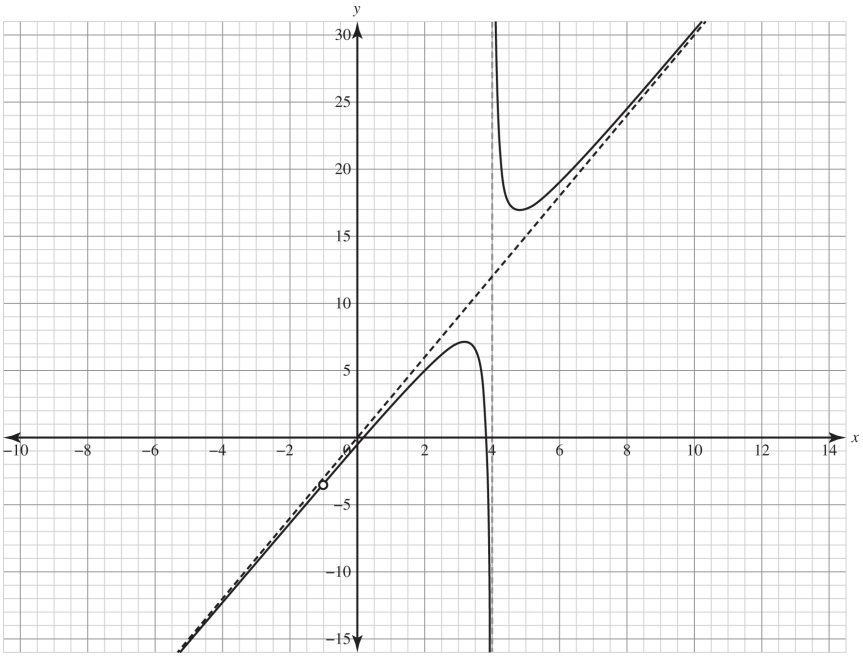


Figure 3.13

3.7 Rational Inequalities

The key to solving rational inequalities comes from the geometric property that the graph of $y = f(x)$ lies above the x-axis when $f(x) > 0$ and below the x-axis when $f(x) < 0$.

Solving Rational Inequalities

1. Write the inequality in one of these forms:

   $f(x) > 0$ $f(x) \geq 0$ $f(x) < 0$ $f(x) £ 0$
2. Determine the critical values of $f(x)$. These are the x-intercepts and vertical asymptotes of the rational expression.
3. Plot the critical values on a number line.
4. Use test points to determine whether the graph of $f(x)$ is above or below the x-axis on each of the intervals determined by the critical values.
5. Find the solutions of the inequality.

› Example

Solve: $\frac{3x}{x+2} \geq 2$

✓ Solution

1. Rewrite the inequality in the form $\frac{f(x)}{g(x)} \geq 0$. Subtract 2 from both sides of the inequality, and rewrite as a single fraction:

$$\frac{3x}{x+2} - 2 \geq 0$$
$$\frac{3x}{x+2} - \frac{2(x+2)}{(x+2)} \geq 0$$
$$\frac{3x - 2x - 4}{x+2} \geq 0$$
$$\frac{x-4}{x+2} \geq 0$$

2. Determine the critical values for the rational expression. The critical values are defined to be the zeros and the vertical asymptotes for the rational expression. In this case, when $x = 4$, the numerator is 0, and when $x = -2$, the denominator is 0.
3. Plot the critical values on a number line. Use a closed circle to indicate that the critical value is included in the solution set, as shown in Figure 3.14. Use an open circle if the number is not included in the solution set, also shown in Figure 3.14.

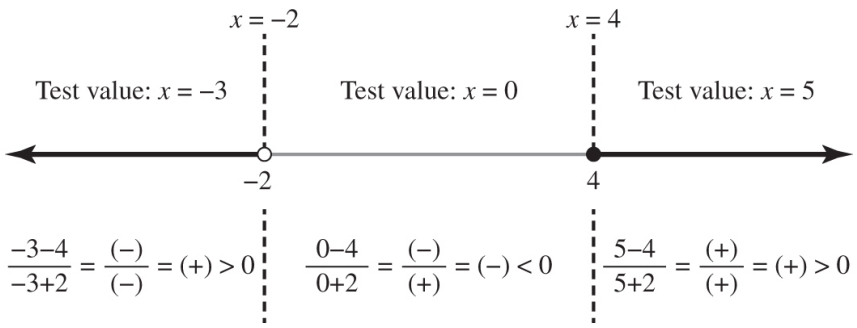


Figure 3.14

4. Choose a test value for x in each of the intervals determined by the critical values. Substitute that test value into the rational expression. Use the substitution to determine the signs of the numerator and denominator as well as their quotient, as shown in Figure 3.14. For example, in the left interval, let $x = -3$ represent the test value. Substitute into the rational expression:

$$\frac{(-3) - 4}{(-3) + 2} = \frac{-7}{-1} = +7 \geq 0$$

5. The solution intervals are $(-\infty, -2) \cup [4, \infty)$.

Example

Solve for x: $\frac{5}{x-3} \leq \frac{2}{x}$

✓ Solution

1. Rewrite the given inequality in the form $\frac{f(x)}{g(x)} \leq 0$, and rewrite as a single fraction:

$$\frac{5}{x-3} - \frac{2}{x} \leq 0$$
$$\frac{5x - 2(x-3)}{x(x-3)} \leq 0$$
$$\frac{3x+6}{x(x-3)} \leq 0$$

2. To solve for the critical values, set the numerator and denominator each equal to 0 and solve for x:

$$3x + 6 = 0 \rightarrow x = -2$$
$$x(x-3) = 0 \rightarrow x = 0, x = 3$$

3. Plot the critical values on a number line. Place a closed circle around −2 since the rational expression can be equal to 0. Use open circles around 0 and 3 since, for these values of x, the rational expression is undefined. This is shown in Figure 3.15.

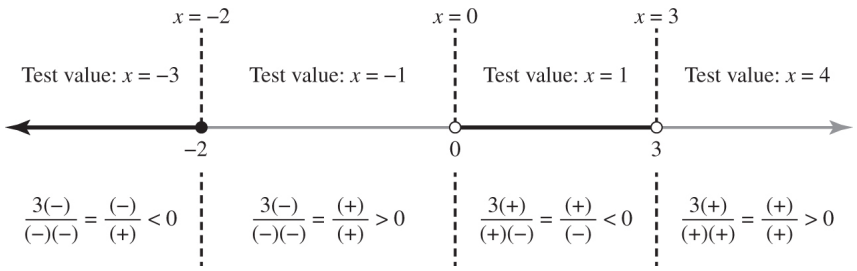


Figure 3.15

4. Determine the sign of the rational expression on each of the four intervals determined by the three critical values, as shown in Figure 3.15.
5. The solution intervals are $(-\infty, -2] \cup (0, 3)$.

Multiple-Choice Questions

1. Which of the following is equivalent to $\lim_{x \to \infty} \frac{4x^2}{x^2 + 5{,}000x}$?

   (A) 0
   (B) $\frac{1}{1{,}250}$
   (C) 1
   (D) 4

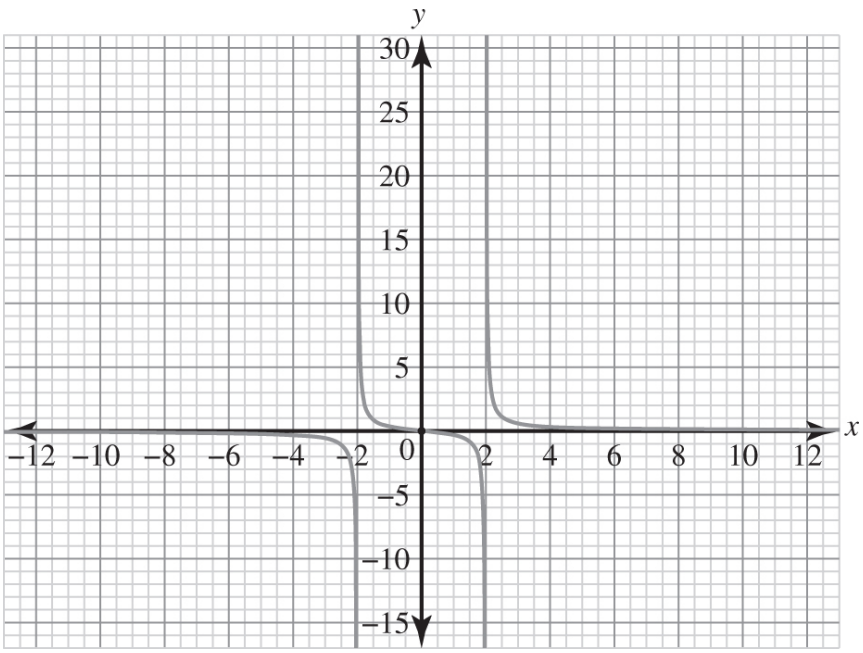
2. The graph of which of the following equations has $y = 1$ as an asymptote?

   (A) $y = \frac{x^2 - 1}{x - 1}$
   (B) $y = \frac{x}{x + 1}$
   (C) $y = \frac{x^2}{x - 1}$
   (D) none of these

3. Which of the following is the end behavior asymptote for the function $f(x) = \frac{x^3 - 8}{x^2 - 3x - 4}$?

   (A) $y = 13x + 4$
   (B) $y = x - 3$
   (C) $y = x + 3$
   (D) $y = x - 11$

4. Which of the following equations could represent the graph shown below?



(A) $y = \frac{x}{x^2 - 4}$

(B) $y = \frac{x^2}{x^2 - 4}$

(C) $y = \frac{x^3}{x^2 - 4}$

(D) $y = \frac{x}{x^2 - 2}$

5. Which of the following functions could represent $f(x)$ if $\lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0^+} f(x) = \infty$?

(A) $f(x) = \frac{1}{x^2}$

(B) $f(x) = \frac{1}{x(x - 3)}$

(C) $f(x) = \frac{1}{x - 1}$

(D) $f(x) = \frac{1}{x}$

6. A rare species of insect was discovered on a tropical island. To protect the species, scientists moved the insects to a protected area. The population P of the insect t months after being moved is $P(t) = \frac{50(1 + 0.5t)}{(2 + 0.01t)}$. How many insects were initially discovered?

(A) 5

(B) 25

(C) 37

(D) 50

7. A rare species of insect was discovered on a tropical island. To protect the species, scientists moved the insects to a protected area. The population P of the insect t months after being moved is $P(t) = \frac{50(1 + 0.5t)}{(2 + 0.01t)}$. What is the largest population that the protected area can sustain?

(A) 5

(B) 50

(C) 2,500

(D) none of these

8. $\lim_{n \to \infty} \frac{-5n + 3n^3}{n^3 - 2n^2 + 1} =$

(A) -5

(B) 1

(C) 3

(D) nonexistent

9. If the graph of $y = \frac{ax + b}{x + c}$ has a horizontal asymptote at $y = 2$ and a vertical asymptote at $x = -3$, what is the sum of a and c?

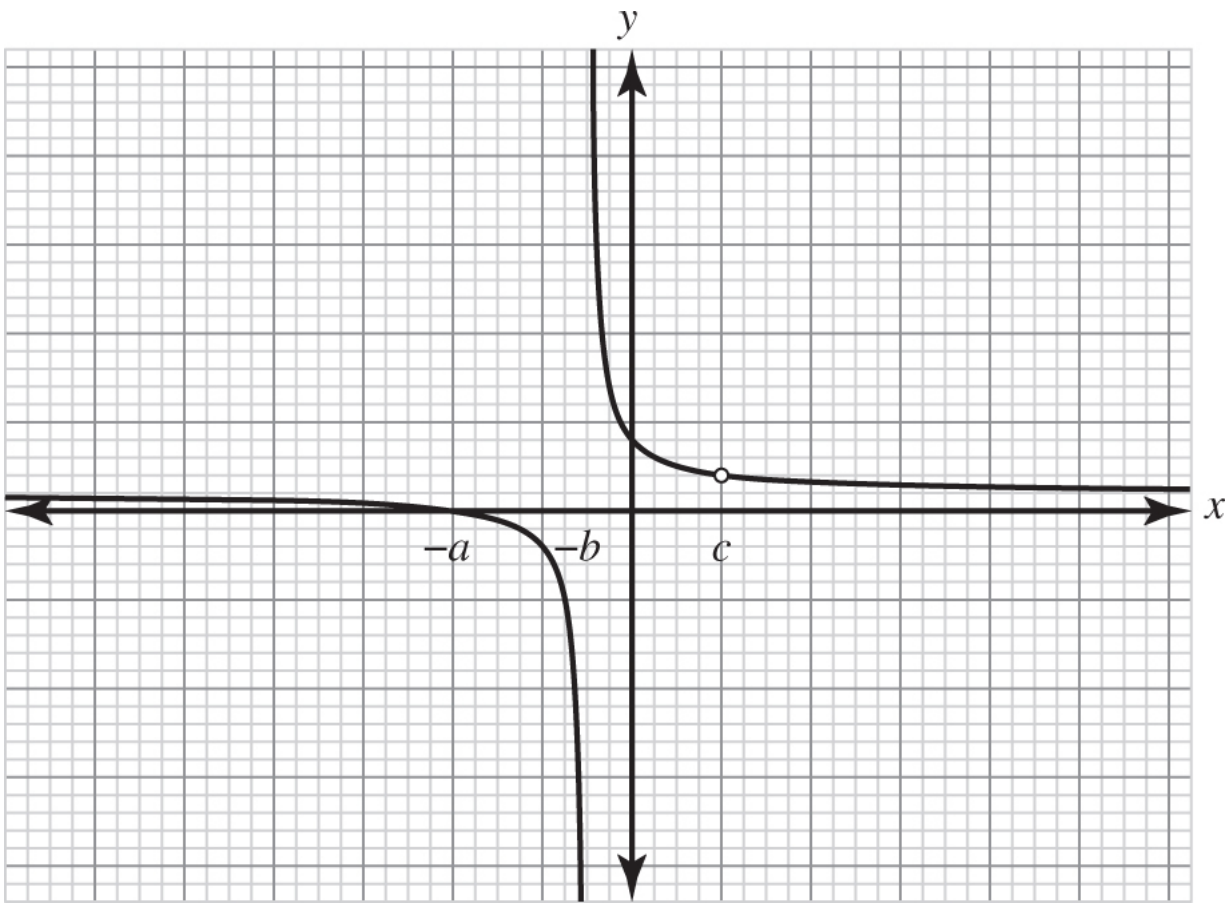
(A) –5

(B) –1

(C) 1

(D) 5

10. Given the graph of the rational function, $c(x)$, which of the following could be its equation?



(A) $c(x) = \frac{(x - c)(x + a)}{(x - c)(x + b)}$

(B) $c(x) = \frac{(x + c)(x - a)}{(x + c)(x - b)}$

(C) $c(x) = \frac{(x + b)(x + a)}{(x + b)(x - c)}$

(D) $c(x) = \frac{(x - b)(x + a)}{(x - b)(x + c)}$

Answer Explanations

1. **(D)** Since the limit has x approaching infinity, this limit is assessing end behavior. To find the end behavior of a rational function, compare the degree of the numerator to the degree of the denominator. Since 2 = 2, the end behavior asymptote is the horizontal asymptote with the equation $y = 4$. Therefore, all the outputs would be approximately 4.

2. (B) For a rational function to have a horizontal asymptote at $y = 1$, the degrees of the numerator and denominator must be the same and the ratio of the leading coefficients of the numerator and denominator must reduce to 1. This happens only with the equation $y = \frac{x}{x+1}$.

3. (C) To find the end behavior asymptote, compare the degree of the numerator to the degree of the denominator. The degree of the numerator is greater than the degree of the denominator, which means there is no horizontal asymptote. Perform polynomial long division to find the end behavior asymptote:

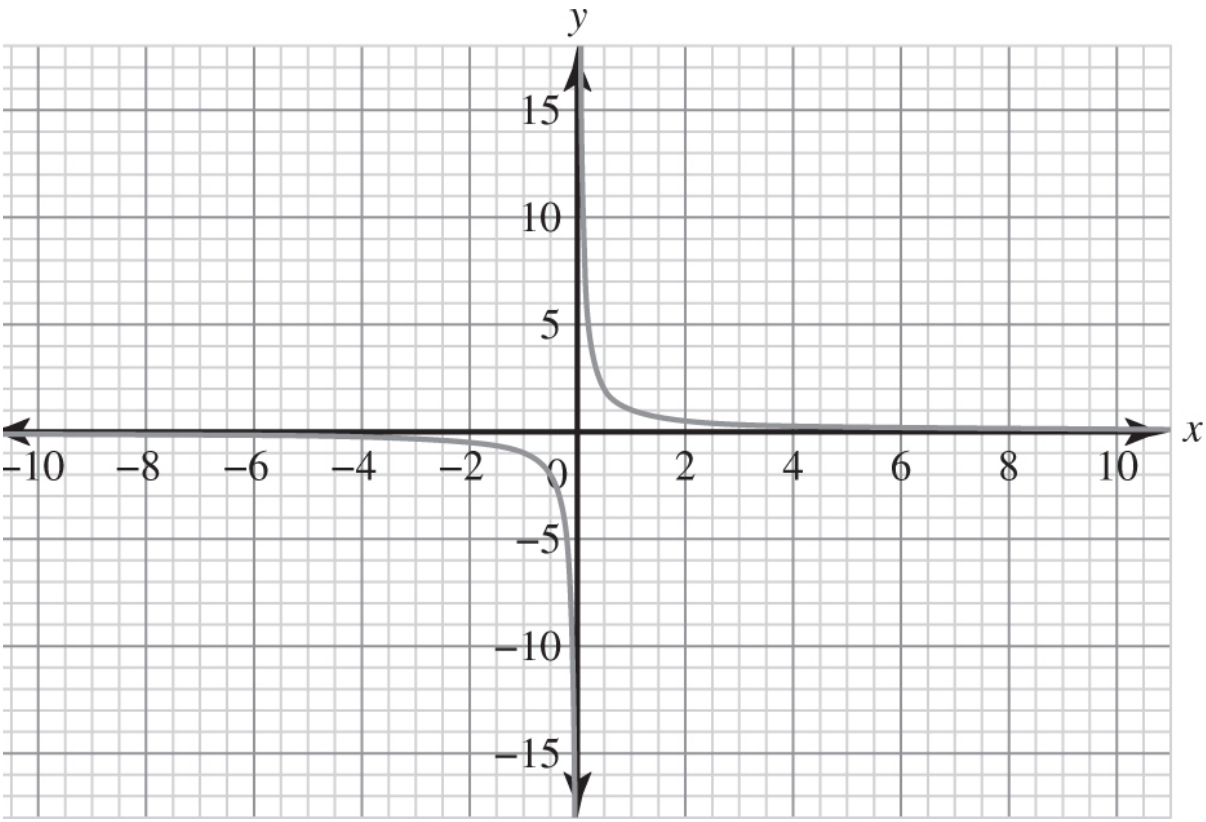
$$\begin{array}{r} x+3 \\ x^2-3x-4\overline{)\,x^3-8} \\ \underline{-(x^3-3x^2-4x)} \\ 3x^2+4x-8 \\ \underline{-(3x^2-9x-12)} \\ 13x+4 \end{array}$$

$$f(x) = \frac{x^3-8}{x^2-3x-4} = x+3+\frac{13x+4}{x^2-3x-4}$$

The end behavior asymptote is a slant asymptote with the equation $y = x + 3$.

4. (A) The graph has vertical asymptotes at $x = -2$ and $x = 2$. The denominator must have zeros that are $x = -2$ and $x = 2$, eliminating Choice (D). The graph has a horizontal asymptote of $y = 0$. The degree of the numerator must be less than the degree of the denominator.

5. (D) The unbounded behavior of the given limits occurs only when there is a vertical asymptote at the value x is approaching. The function must have a vertical asymptote at $x = 0$, which eliminates Choice (C). The graphs of Choices (A) and (B) approach positive infinity as x approaches 0 from the left. Therefore, both of those choices need to be eliminated since the question is asking for a graph to approach negative infinity as x approaches 0 from the left. The graph of $f(x) = \frac{1}{x}$ that follows satisfies the given limits.



6. (B) The initial population can be determined by evaluating $P(0)$:

$$P(0) = \frac{50(1 + 0.5(0))}{(2 + 0.01(0))} = \frac{50}{2} = 25$$

7. (C) Determining the largest population that the area can sustain refers to the end behavior of the rational function. To determine end behavior, compare the degree of the numerator to the degree of the denominator. Since the degrees of the numerator and denominator both equal 1, the end behavior is a horizontal asymptote that is the ratio of the leading coefficients. To determine the leading coefficients, multiply the numerator by the factored-out constant:

$$P(t) = \frac{50(1 + 0.5t)}{(2 + 0.01t)} = \frac{50 + 25t}{2 + 0.01t}$$

The horizontal asymptote is $y = \frac{25}{0.01} = 2{,}500$.

8. (C) Evaluating infinite limits of rational functions can be determined by their end behavior. To find the end behavior, compare the degree of the numerator to the degree of the denominator. The degree of the

numerator and denominator both equal 3. Therefore, the end behavior follows the horizontal asymptote. Its equation is determined by taking the ratio of the leading coefficients:

$$y = \frac{3}{1} = 3$$

9. **(D)** Since the rational function has a vertical asymptote at $x = -3$, its denominator must have a zero of $x = -3$. Therefore, the denominator must be the factor $x + 3$, so $c = 3$. Since the rational function has a horizontal asymptote at $y = 2$, the ratio of the leading coefficients must also be 2. Therefore, $y = 2 = \frac{a}{1}$ and $a = 2$. The sum of a and c is $2 + 3 = 5$.

10. **(A)** The graph has an x-intercept at $-a$, meaning there is a zero at $x = -a$. So there must be a factor of $(x - (-a)) = (x + a)$ in the numerator of the function. There is a vertical asymptote at $x = -b$, so there must be a factor of $(x - (-b)) = (x + b)$ in the denominator. There is a hole at $x = c$, so there must be a common factor in the numerator and denominator of $(x - c)$.

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4

Parent Functions

Learning Objectives

In this chapter, you will learn:

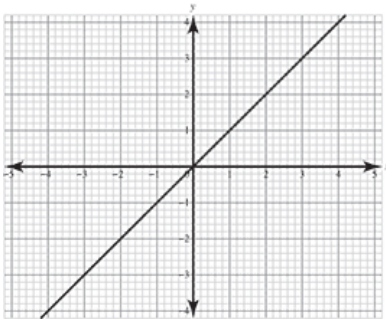
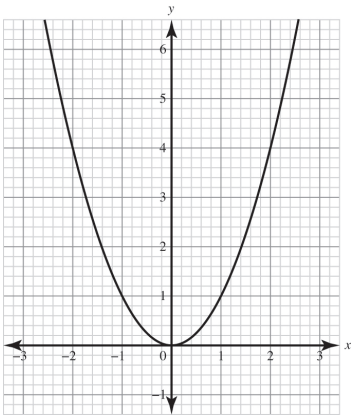
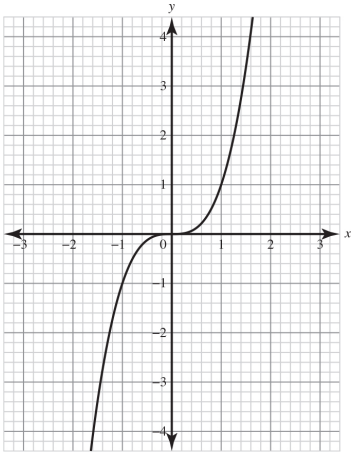
- ➜ The different properties and graphs of parent functions
- ➜ How to transform the graphs of parent functions
- ➜ How to model scenarios using functions
- ➜ How to compose functions
- ➜ How to find the inverse of a function

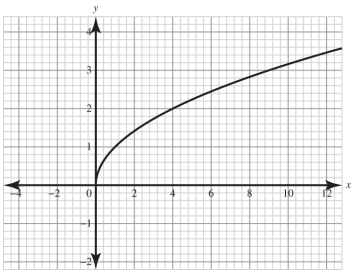
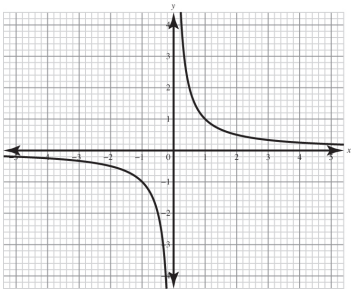
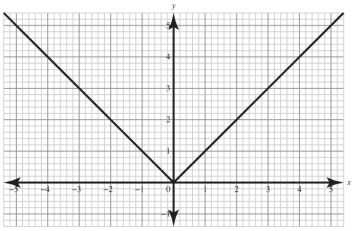
In previous chapters, we have seen that linear functions are represented graphically by lines and quadratic functions are represented by parabolas. Each function in its most basic form has a corresponding graph that can be changed based upon transformations to its basic function, also called a parent function.

4.1 Graphs and Properties of Basic Functions

Table 4.1 discusses the graphs and properties of basic parent functions. Other special functions will be discussed in their respective units.

Table 4.1

Parent Function	Equation	Graph	Properties
Linear	$f(x) = x$		■ Domain and range is the set of real numbers ■ Odd function; has origin symmetry ■ Includes point (0, 0) ■ Increases throughout its domain
Quadratic	$f(x) = x^2$		■ Domain and range is the set of real numbers ■ Even function; has *y*-axis symmetry ■ Includes point (0, 0) ■ Decreases then increases ■ Has a relative extremum
Cubic	$f(x) = x^3$		■ Domain and range is the set of real numbers ■ Odd function; has origin symmetry ■ Includes point (0, 0) ■ Increases throughout its domain ■ Has a point of inflection

Parent Function	Equation	Graph	Properties
Square root	$f(x) = \sqrt{x}$		■ Domain and range is the set of all nonnegative real numbers ■ Neither even nor odd function ■ Includes point (0, 0) ■ Increases throughout its domain
Reciprocal	$f(x) = \frac{1}{x}$		■ Domain and range is the set of all real numbers except 0 ■ Odd function; symmetric with respect to the origin ■ *x*- and *y*-axes are asymptotes ■ Decreases on the intervals $(-\infty, 0)$ and $(0, \infty)$
Absolute value	$f(x) = \lvert x \rvert$		■ Domain and range is the set of real numbers ■ Even function; symmetric with respect to the *y*-axis ■ Includes point (0, 0)

Parent Function	Equation	Graph	Properties
			■ Decreases then increases

Piecewise-Defined Functions

A function may change its definition over its domain. The absolute value function, $f(x) = |x|$, is an example of a piecewise-defined function. When $x < 0$, $f(x) = -x$, and when $x \geq 0$, $f(x) = x$. This can also be written as:

$$f(x) = |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

To graph a piecewise function, it is important to check the inequalities associated with each function and where the boundaries change. Making a table of values can be helpful.

› Example

Graph the piecewise function: $f(x) = \begin{cases} x^2, x < 2 \\ x + 2, 2 \leq x < 4 \\ \sqrt{x}, x \geq 4 \end{cases}$

✓ Solution

Three different functions make up $f(x)$. To help organize, create a table of values, as shown in Table 4.2. Pay attention to the boundaries of the inequality. At each boundary point, the two different functions should be evaluated. The values in bold are points that lie on the function.

Table 4.2

x	0	1	2	3	4	9
$f(x)$	$\mathbf{x^2}$ $(0)^2 = 0$	$\mathbf{x^2}$ $(1)^2 = 1$	x^2 $(2)^2 = 4$	$\mathbf{x + 2}$	$\mathbf{x + 2}$	$\sqrt{x}$ $\sqrt{9} = 3$

			$x + 2$ (2) + 2 = 4	(3) + 2 = 5	(4) + 2 = 6 $\sqrt{x}$ $\sqrt{4} = 2$	

When graphing, all the points in bold should be plotted on the graph. At the boundary points, graph up to that point. However, plot the boundary point with an open circle if it is not part of the interval. An open circle shows that the graph goes up to that point but does not include it.
The graph of $f(x)$ is shown in Figure 4.1. At $x = 2$, the output values for both functions are the same. So while the graph of the quadratic would have an open circle at (2, 4), the graph of the line has a closed circle at (2, 4), closing the open circle. Also note that at $x = 4$, the two functions have different output values. The graph of the line goes up to but does not include the point (4, 6), which is why it is graphed with an open circle. The graph drops down to (4, 2) for the square root and is closed there. It is important to note that often piecewise functions are not continuous. At $x = 4$ there is a discontinuity, specifically called a jump discontinuity since you must lift your pencil to jump from one graph to the other. These discontinuities often occur at the boundaries of the different intervals of each piece.

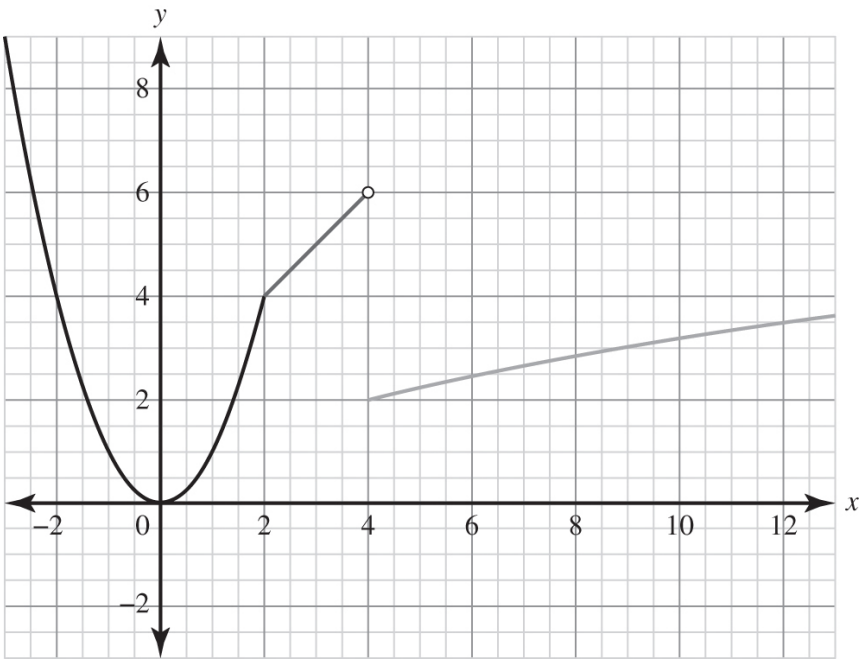


Figure 4.1

4.2 Transformations of Functions

A transformation of a function moves each point of the graph of the function according to a given rule. Flipping a graph over a line is a reflection over that line. Shifting a graph vertically, horizontally, or both vertically and horizontally is a translation. Reflections and translations do not affect the size or shape of the graph. Stretching or shrinking a graph is known as a dilation and does affect the size or shape of the graph. Each of these transformations has a corresponding additive and/or multiplicative rule that can be performed on the function. If two functions are related to each other by a simple transformation, knowing the graph of one function allows you to determine the graph of the transformed function.

Vertical Translations

Adding a number k to a function shifts the graph up if k is positive and shifts it down if k is negative.

Vertical Translation of Function $f(x)$

Given the function $f(x)$:

The function $g(x) = f(x) + k$ is an additive transformation of the function $f(x)$ that results in a vertical translation of the graph of f by k units.

Example

Given $f(x) = x^2$, describe the transformations of $g(x)$ and $h(x)$ as they relate to $f(x)$. Graph each to confirm your answer.

1. $g(x) = x^2 + 3$
2. $h(x) = x^2 - 4$

Solution

The parent function for each is $f(x) = x^2$, which graphs a parabola that contains the point (0, 0).

1. The function $g(x)$ is based upon $f(x)$ but with an addition of 3. By adding 3 to the function, the graph of $f(x)$ shifts up 3 units. This can easily be seen in Figure 4.2 as the point (0, 0) moves to (0, 3). All the points in the graph move up 3 units.

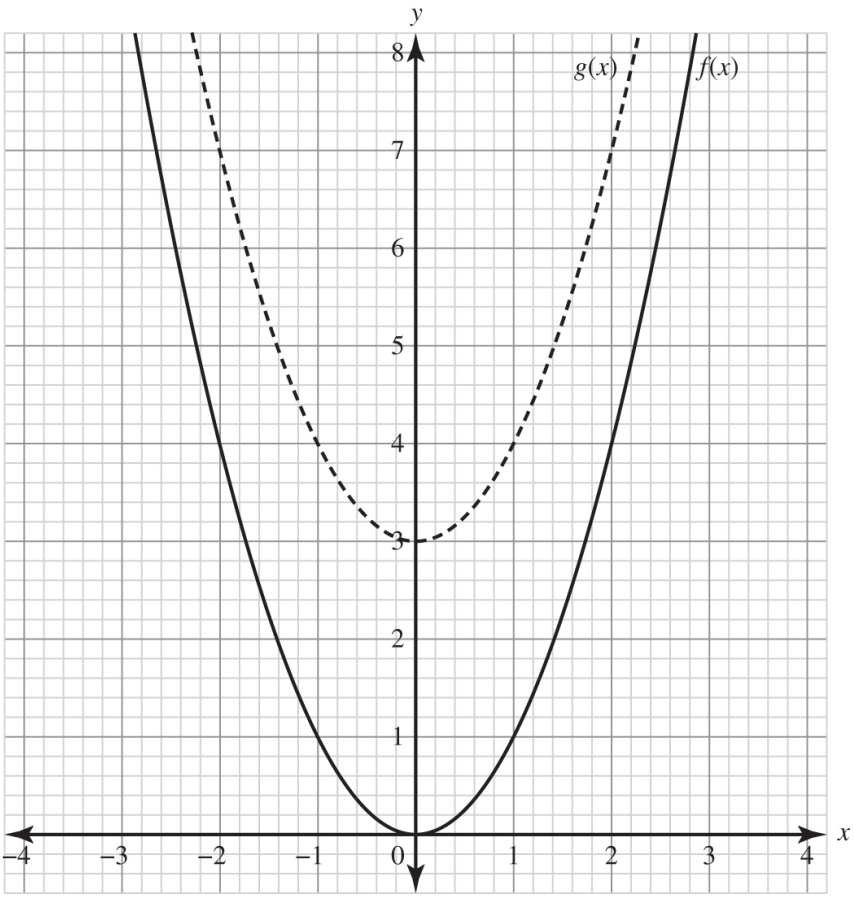


Figure 4.2

2. The function $h(x)$ is based upon $f(x)$ but with a subtraction of 4. By subtracting the function by 4, the graph of $f(x)$ shifts 4 units down. This can easily be seen in Figure 4.3 as the point (0, 0) moves to (0, –4). All the points in the graph move down 4 units.

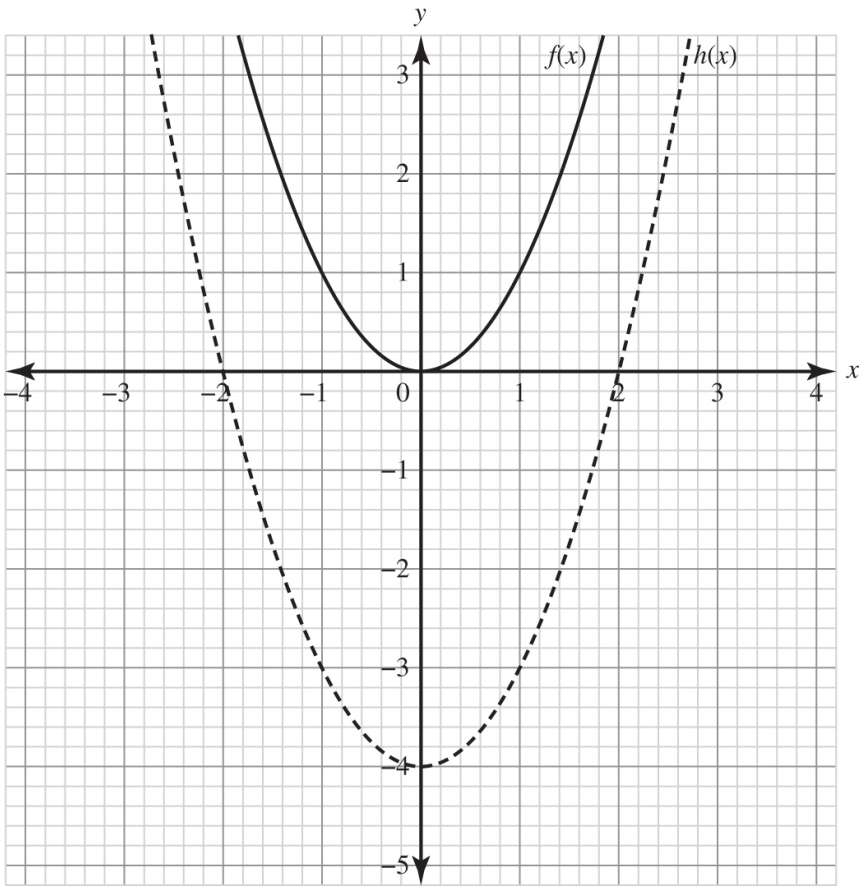


Figure 4.3

Horizontal Translations

Subtracting a number h from x in a function shifts the graph to the right if h is positive and shifts it to the left if h is negative.

Horizontal Translation of Function $f(x)$

Given the function $f(x)$:

The function $g(x) = f(x + h)$ is an additive transformation of the function $f(x)$ that results in a horizontal translation of the graph of f by $-h$ units.

Example

Given $f(x) = |x|$, describe the transformations of $g(x)$ and $h(x)$ as they relate to $f(x)$. Graph each to confirm your answer.

1. $g(x) = |x + 3|$
2. $h(x) = |x - 4|$

☑ Solution

The parent function for each is $f(x) = |x|$, which graphs the piecewise function made up of $y = -x$ and $y = x$ and contains the point (0, 0).

1. In $g(x)$, adding 3 to x shifts the graph of $f(x)$ 3 units left. This can easily be seen in Figure 4.4 as the point (0, 0) moves to (–3, 0). All the points in the graph move left 3 units.

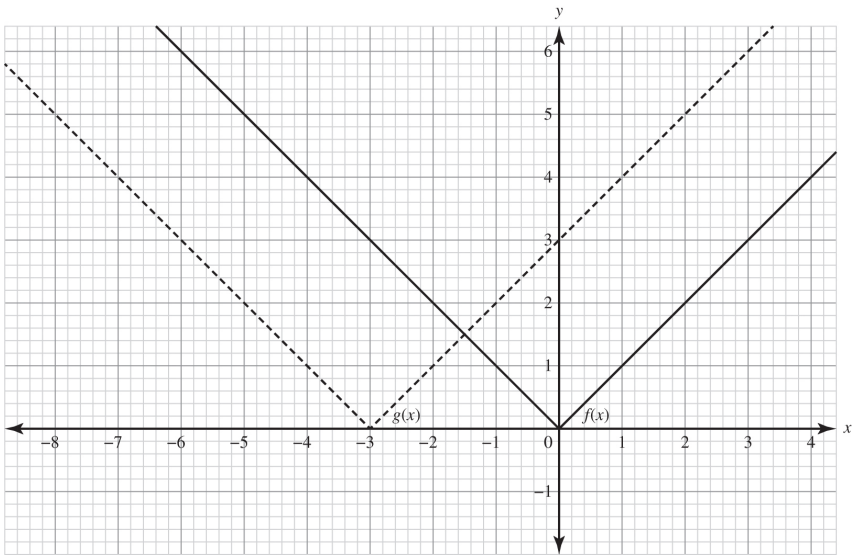


Figure 4.4

2. In $h(x)$, subtracting 4 from x shifts the graph of $f(x)$ 4 units right. This can easily be seen in Figure 4.5 as the point (0, 0) moves to (4, 0). All the points in the graph move right 4 units.

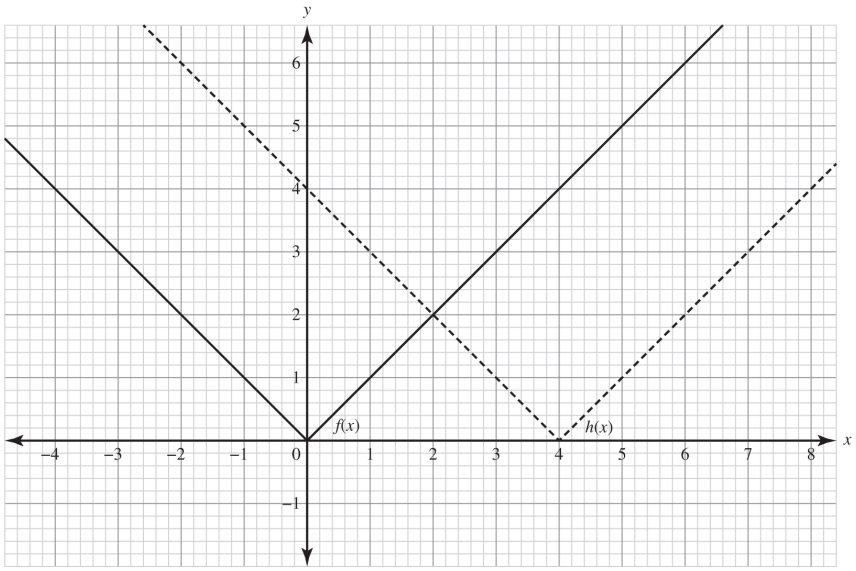


Figure 4.5

Vertical Dilation and Reflection over *x*-axis

Multiplying a positive number, a, by a function, where $a \neq 0$, vertically stretches or compresses the graph of the function by a factor of a.

- If $a > 1$, the points of the function move farther away from the x-axis.
- If $0 < a < 1$, the points of the function move closer to the x-axis.

If the number being multiplied to the function is negative, the graph reflects over the x-axis.

Vertical Dilation and Reflection of Function *f*(*x*)

Given the function $f(x)$:

The function $g(x) = af(x)$, where $a \neq 0$, is a multiplicative transformation of the function $f(x)$ that results in a vertical dilation of the graph of f by a factor of $|a|$. If $a < 0$, the transformation involves a reflection over the x-axis.

Example

Given $f(x) = x^2$, describe the transformations of $g(x)$, $h(x)$, and $i(x)$ as they relate to $f(x)$. Graph each to confirm your answer.

1. $g(x) = 2x^2$
2. $h(x) = \frac{1}{2}x^2$
3. $i(x) = -3x^2$

☑ Solution

The parent function for each is $f(x) = x^2$, which graphs a parabola that contains the point (1, 1).

1. By multiplying the function by 2, the graph of $f(x)$ vertically dilates. All the x-values remain fixed, and their corresponding y-values are multiplied by 2. This can easily be seen in Figure 4.6 as the point (1, 1) moves to (1, 2). The points on the dilated graph move farther away from the x-axis.

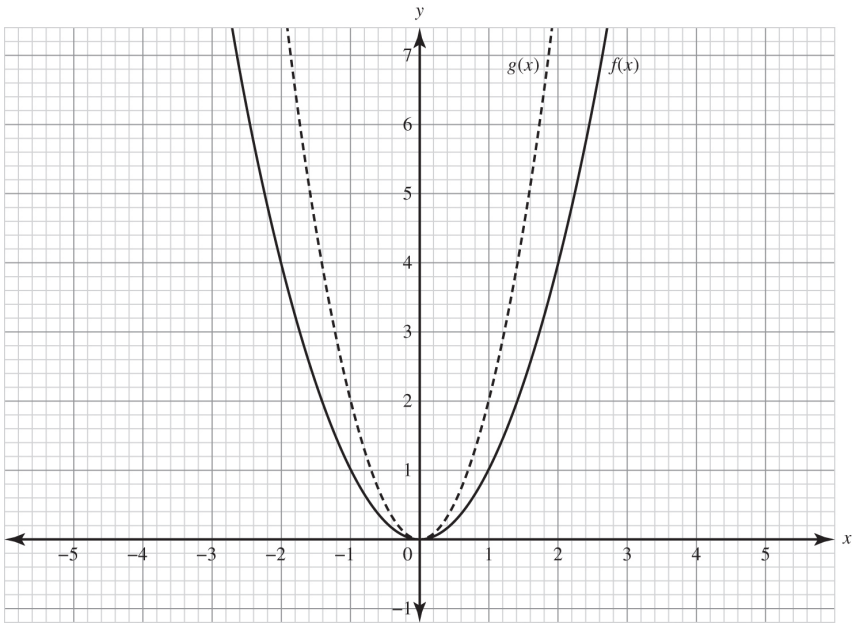


Figure 4.6

2. By multiplying the function by $\frac{1}{2}$, the graph of $f(x)$ vertically dilates. All the x-values remain fixed, and their corresponding y-values are

multiplied by $\frac{1}{2}$. This can easily be seen in Figure 4.7 as the point (1, 1) moves to $(1, \frac{1}{2})$. The points on the dilated graph move closer to the *x*-axis.

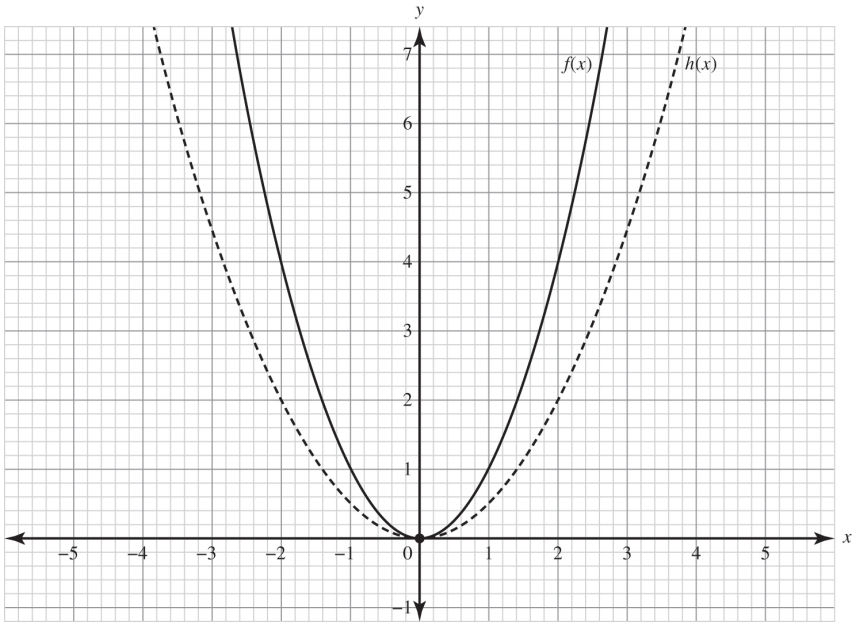


Figure 4.7

3. By multiplying the function by –3, the graph of $f(x)$ vertically dilates by a factor of $|-3|$ and is reflected over the *x*-axis. All the *x*-values remain fixed, and their corresponding *y*-values are multiplied by –3. This can easily be seen in Figure 4.8 as the point (1, 1) moves to (1, –3). The points on the dilated graph move farther away from the *x*-axis and are also reflected over the *x*-axis.

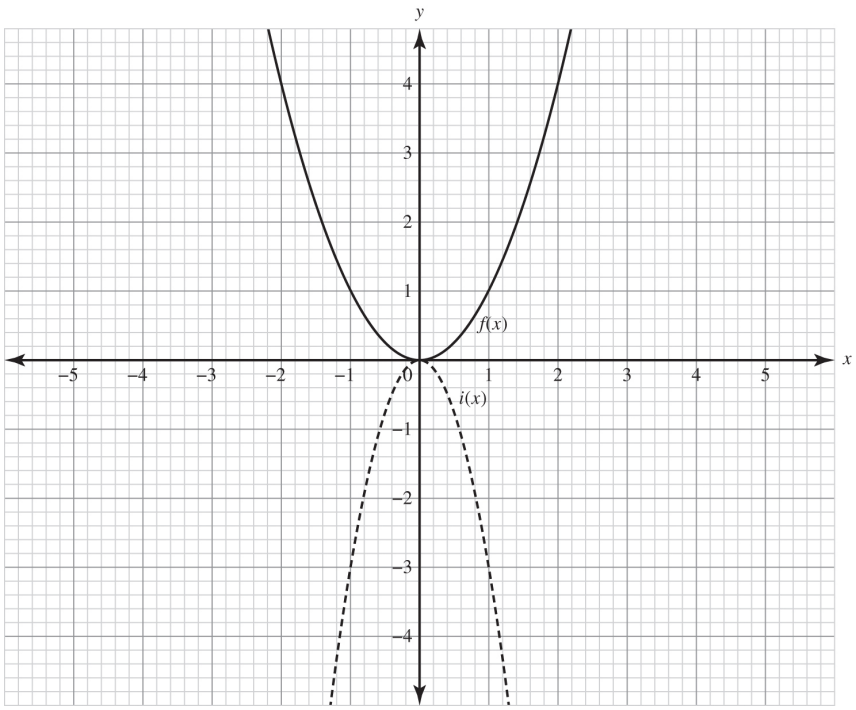


Figure 4.8

Horizontal Dilation and Reflection over *y*-axis

Multiplying a positive number, b, by x in a function, where $b \neq 0$, horizontally stretches or compresses the graph of the function by a factor of $\frac{1}{b}$.

- If $b > 1$, the points of the function move closer to the y-axis.
- If $0 < b < 1$, the points of the function move farther away from the y-axis.

If the number being multiplied by x in the function is negative, the graph reflects over the y-axis.

Horizontal Dilation and Reflection of Function *f*(*x*)

Given the function $f(x)$:

The function $g(x) = f(bx)$, where $b \neq 0$, is a multiplicative transformation of the function $f(x)$ that results in a horizontal dilation of the graph of f by a factor of $\left|\frac{1}{b}\right|$. If $b <$

0, the transformation involves a reflection over the *y*-axis.

Example

Given $f(x) = x^3$, describe the transformations of $g(x)$, $h(x)$, and $i(x)$ as they relate to $f(x)$. Graph each to confirm your answer.

1. $g(x) = (2x)^3$
2. $h(x) = \left(\frac{1}{2}x\right)^3$
2. $i(x) = (-2x)^3$

Solution

The parent function for each is $f(x) = x^3$, which graphs a cubic polynomial that contains the point (1, 1).

1. By multiplying the x of the function by 2, the graph of $f(x)$ horizontally dilates. This isn't the same as a vertical dilation where the output values are scaled; here the input values are scaled. The x in the original $f(x)$ becomes $8x$ in $g(x)$, meaning $g(x)$ reaches a given input value 8 times faster than $f(x)$. This can easily be seen in Figure 4.9 as the point (1, 1) moves to (1, 8).

   Another way to find the points is by using the factor $\left|\frac{1}{2}\right|$. All the *y*-values remain fixed on the parent function, and their corresponding *x*-values are multiplied by $\left|\frac{1}{2}\right|$. The graph of $f(x) = x^3$ contains the point (2, 8). Multiplying the input value 2 by the scale factor $\left|\frac{1}{2}\right|$ simplifies to 1. So the point on the graph of $g(x) = (2x)^3$ will be (1, 8). The point (4, 64) lies on the graph of $f(x) = x^3$, so the point $\left(4 \cdot \frac{1}{2}, 64\right) = (2, 64)$ will lie on the graph of $g(x) = (2x)^3$. As shown in Figure 4.9, the points on the dilated graph move closer to the *y*-axis.

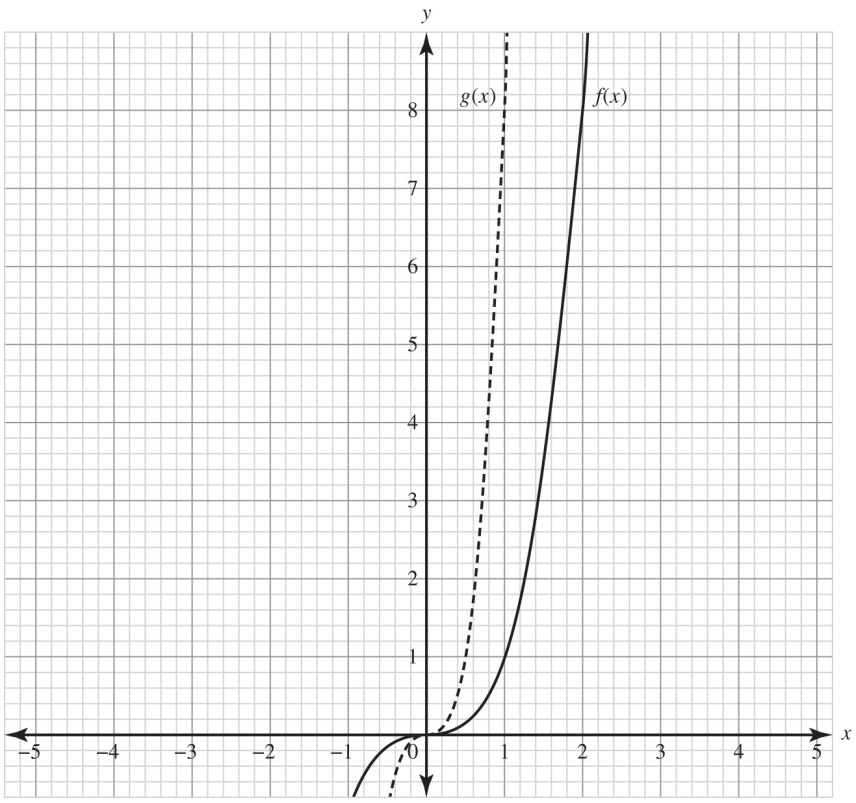


Figure 4.9

2. By multiplying the x of the function by $\frac{1}{2}$, the graph of $f(x)$ horizontally dilates. This isn't the same as a vertical dilation where the output values are scaled; here the input values are scaled. The x in the original $f(x)$ becomes $\frac{1}{8}x$ in $h(x)$. This can easily be seen in Figure 4.10 as the point (1, 1) moves to $\left(1, \frac{1}{8}\right)$.

   Another way to find the points is by using the factor $\left|\frac{1}{\frac{1}{2}}\right| = |2|$. All the y-values remain fixed on the parent function, and their corresponding x-values are multiplied by |2|. The graph of $f(x) = x^3$ contains the point (2, 8). Multiplying the input value 2 by the scale factor |2| simplifies to 4. So the point on the graph of $h(x) = \left(\frac{1}{2}x\right)^3$ will be (4, 8). The point (4, 64) lies on the graph of $f(x) = x^3$, so the point $(4 \cdot 2, 64) = (8,64)$ lies on the graph of $h(x) = \left(\frac{1}{2}x\right)^3$. As shown in Figure 4.10, the points on the dilated graph move farther away from the y-axis.

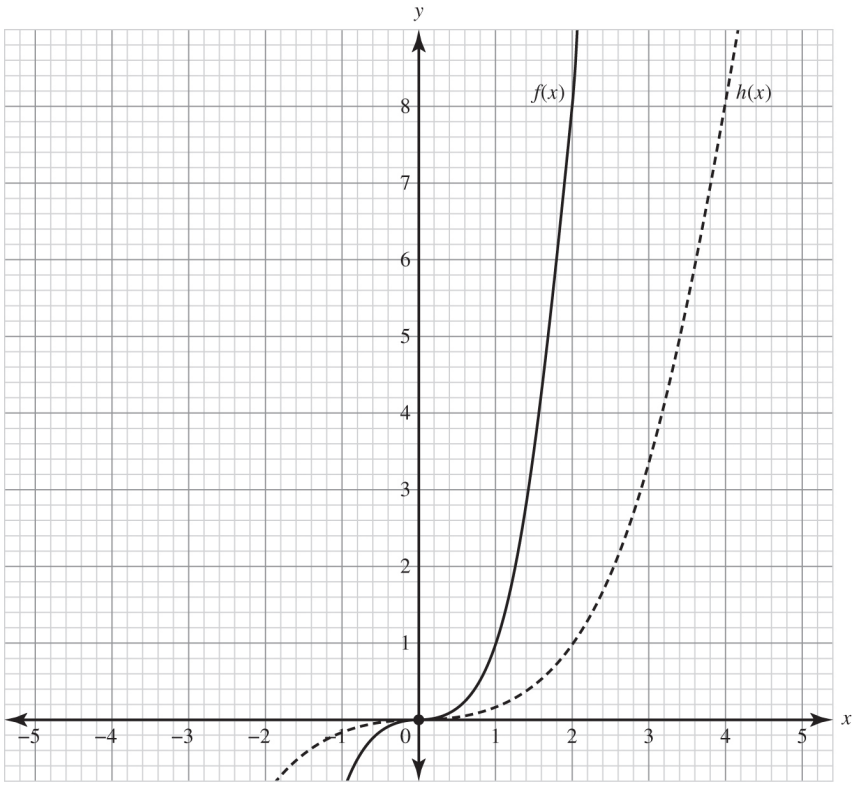


Figure 4.10

3. Function $i(x)$ is the same as the function $g(x)$ in Question 1 except that the input is negative. The graph of $i(x)$ will be the graph of $g(x)$ reflected over the y-axis, as shown in Figure 4.11.

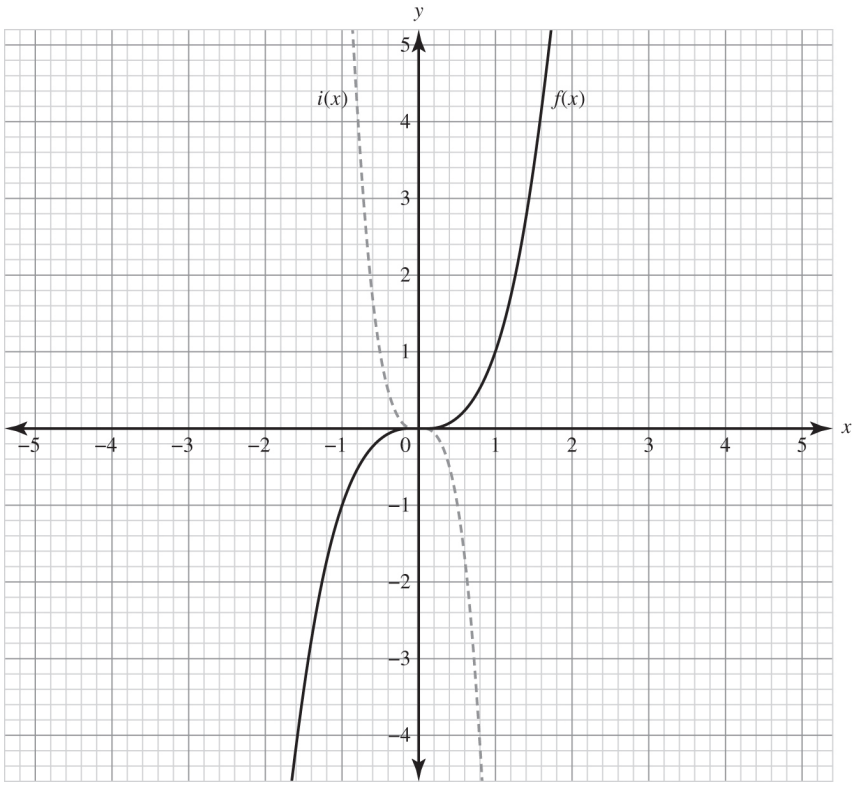


Figure 4.11

Combinations of Transformations

Additive and multiplicative transformations can be combined, resulting in combinations of horizontal and vertical translations, dilations, and reflections. It is often necessary to perform transformations in a certain order to guarantee the arrival at the correct graph. The following steps will help to ensure the correct order of operations.

Steps for a Sequence of Transformations

Apply the following steps when graphing a function containing more than one transformation.

1. Start with the parentheses. Look for a possible horizontal shift.
2. Deal with horizontal or vertical dilations.

3. Deal with reflections over the x- and/or y-axes.
4. Deal with vertical shifts.

This is very similar to the algebraic order of operations known as PEMDAS, which stands for Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction.

Given the function $y = f(x)$, a combination of transformations can be expressed algebraically as:

$$y = af(b(x + c)) + d$$

where
a = vertical stretch or compression
b = horizontal stretch or compression
c = horizontal shift
d = vertical shift

A function that is made up of a combination of transformations is also referred to as a composite function.

Example

The graph of $y = f(x)$ is shown in Figure 4.12 below. Determine the graph of the composite function $y = 2f(x + 1) - 3$ by showing the effect of a sequence of transformations on the graph of $y = f(x)$.

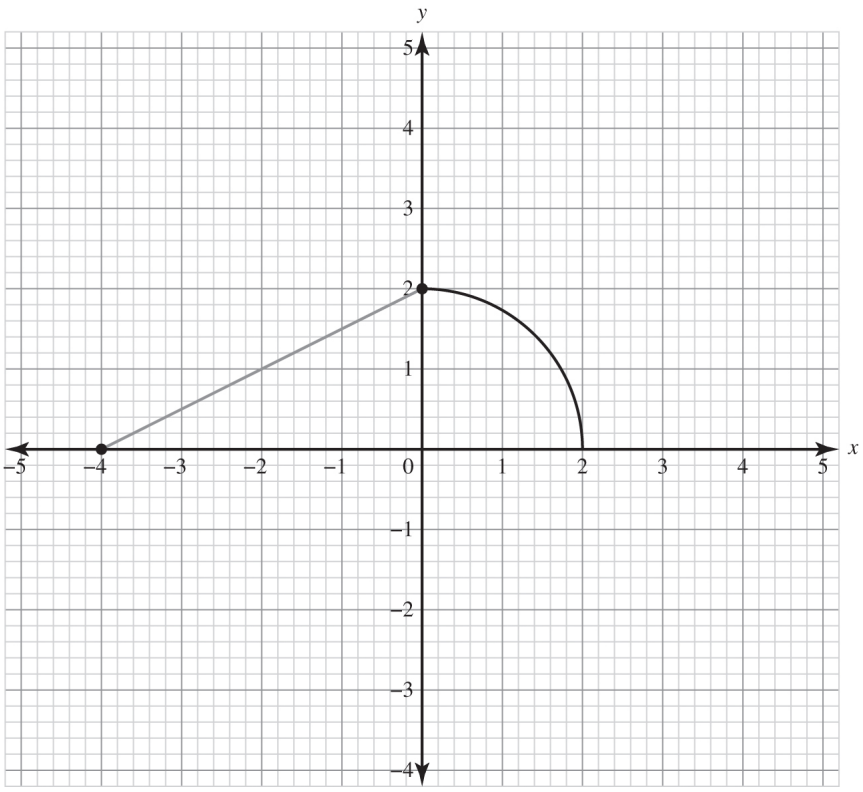


Figure 4.12

☑ Solution

1. Starting with the parentheses, there is a horizontal translation 1 unit to the left, as shown in Figure 4.13.

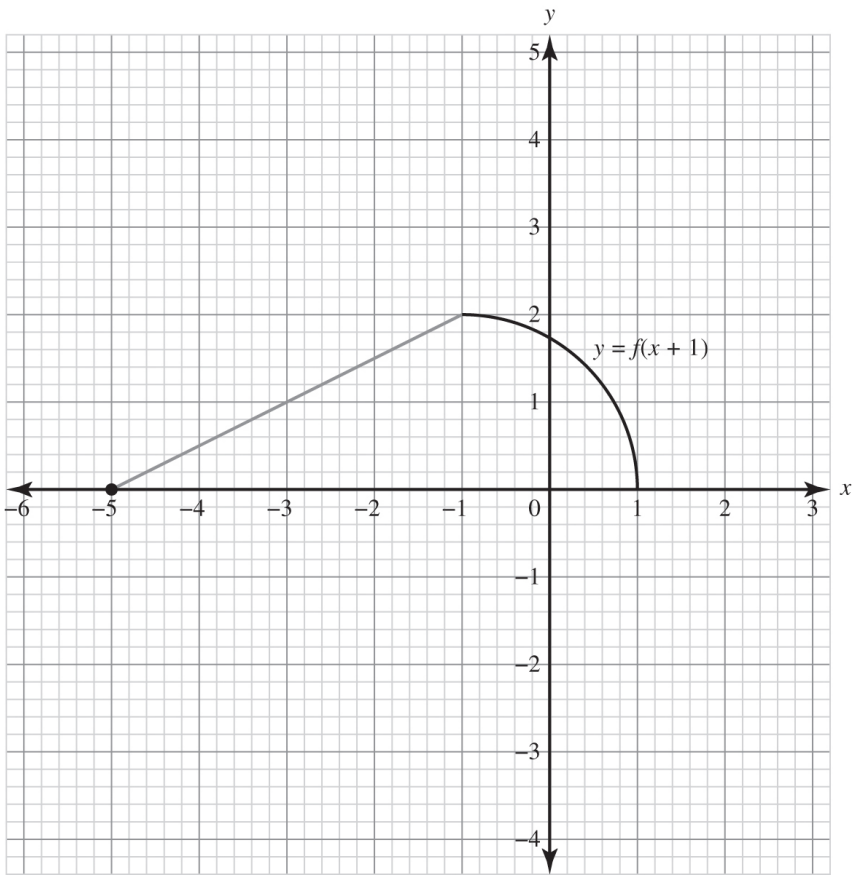


Figure 4.13

2. Next deal with the multiplication. Figure 4.14 shows a vertical dilation by a factor of 2.

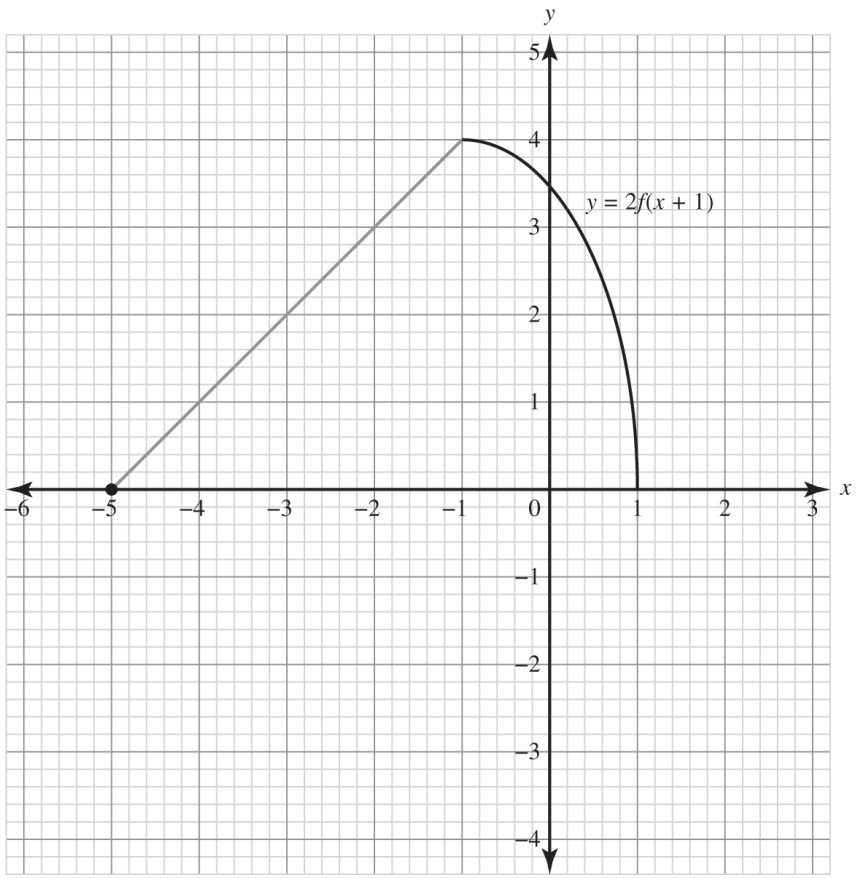


Figure 4.14

3. Finally, deal with the addition/subtraction. There is a vertical translation down 3, as shown in Figure 4.15.

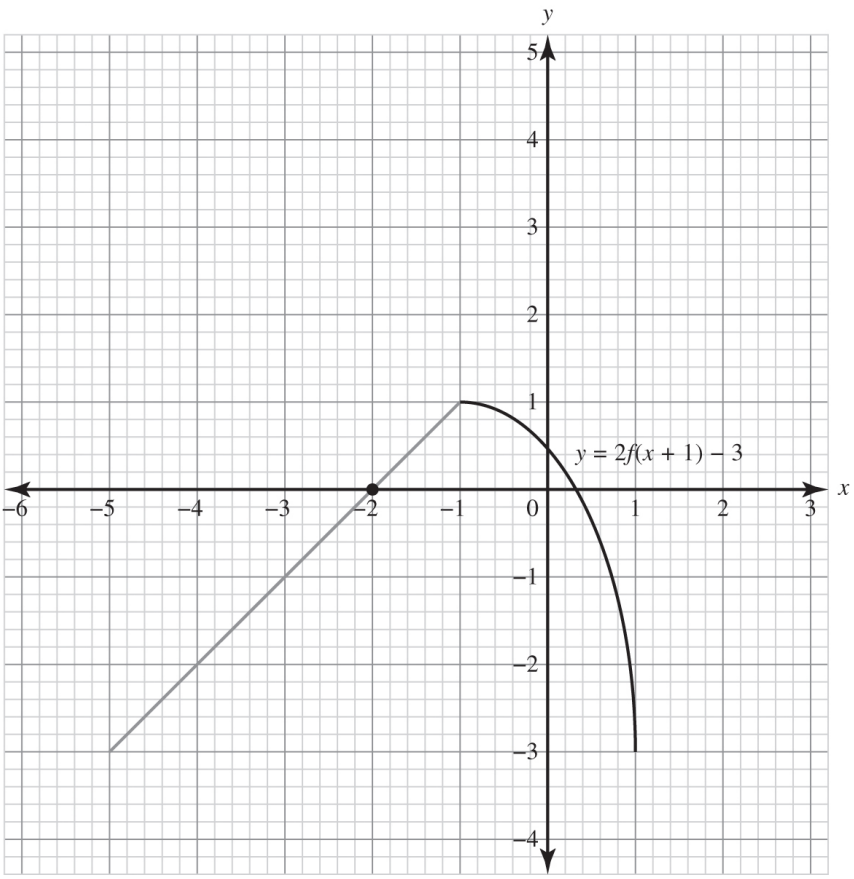


Figure 4.15

Example

The graph of $y = |x|$ undergoes the following transformations, in order:

- A horizontal shift 2 units to the right
- A vertical stretch by a factor of $\frac{1}{3}$
- A vertical translation 7 units up

Find the equation of the graph that results from the above transformations.

Solution

Apply the transformations in the given order:

$$y = |x| \rightarrow y = |x - 2| \rightarrow y = \frac{1}{3}|x - 2| \rightarrow y = \frac{1}{3}|x - 2| + 7$$

Example

Let C_1 be the curve defined by $y_1 = f(x) = x_3 - 16x$. Find the equation of C_2, which is a horizontal shrink of C_1 by a factor of $\frac{1}{2}$ and is reflected over the x-axis.

Solution

Let the equation for C_2 be denoted by y_2. Then:

$$y_2 = -f\left(\frac{x}{\frac{1}{2}}\right) = -f(2x) = -((2x)^3 - 16(2x)) = -(8x^3 - 32x) = -8x^3 + 32x$$

All these different types of transformations alter the graph in certain ways, either by changing their position or their size. Therefore, it is common to find that the domain and range of a function that is a transformation of a parent function may be different from those of the parent function.

Example

Given each of the following, describe a basic graph and a sequence of transformations that can be used to produce a graph of the given function. State the domain and range of the given function and its parent function.

1. $y = (3x)^2 - 4$
2. $y = -3\sqrt{x+1}$

Solution

1. The parent function is $y = x^2$, as shown in Figure 4.16. The domain of $y = x^2$ is all real numbers, and the range is $y \geq 0$.

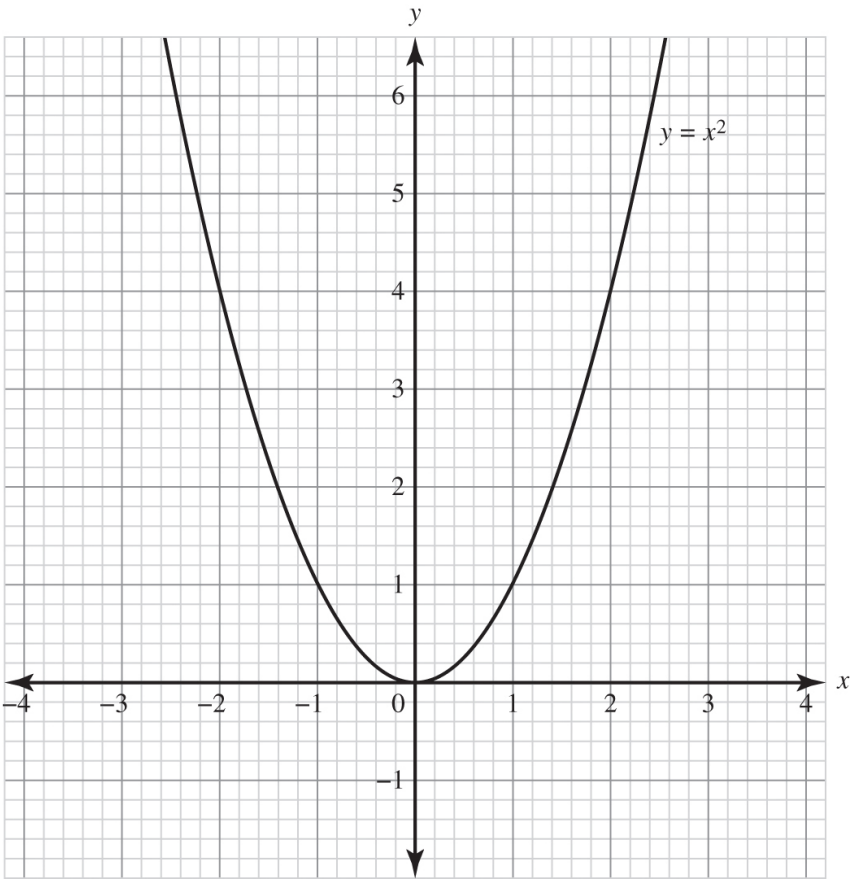


Figure 4.16

Starting with the parentheses, there is a horizontal dilation with a factor of $\frac{1}{3}$. Then there is a vertical shift down 4 units. The result is graphed in Figure 4.17.

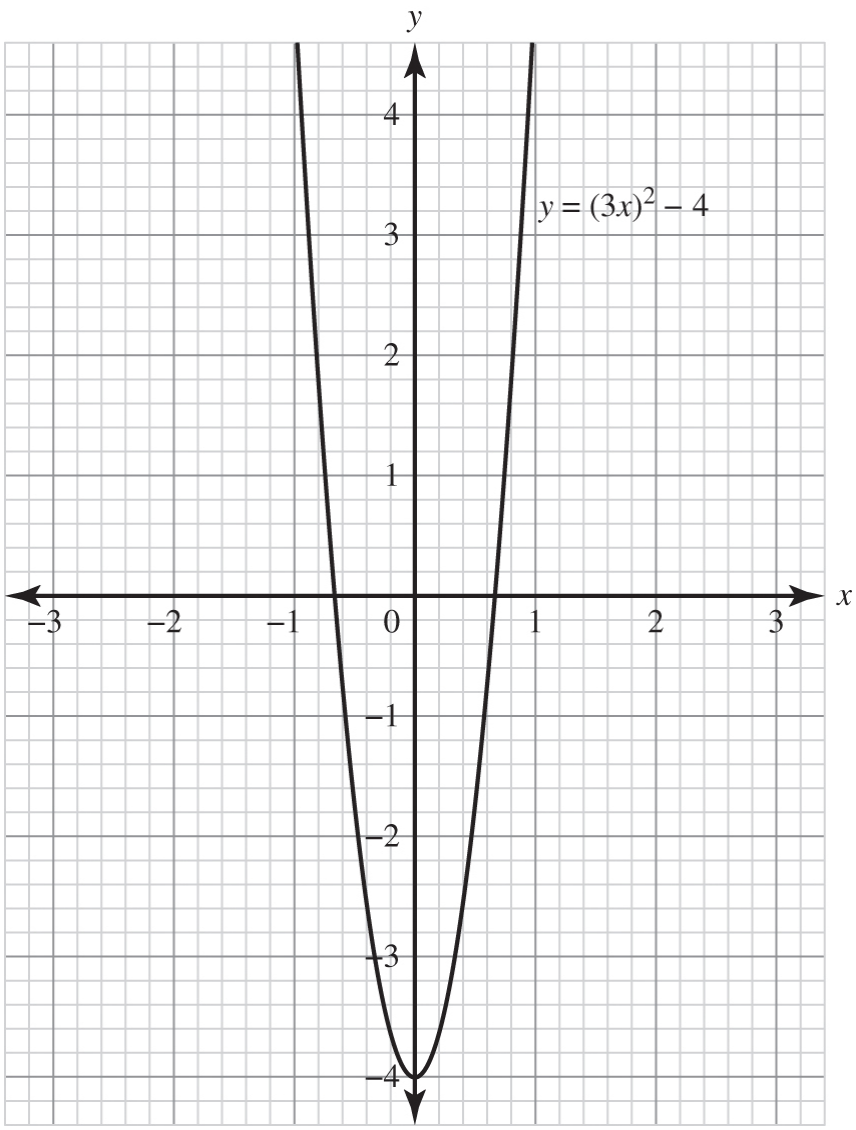


Figure 4.17

The domain of $y = (3x)^2 - 4$ is all real numbers, and the range is $y \geq -4$.

2. The parent function is $y = \sqrt{x}$, as shown in Figure 4.18. The domain of $y = \sqrt{x}$ is $x \geq 0$, and the range is $y \geq 0$.

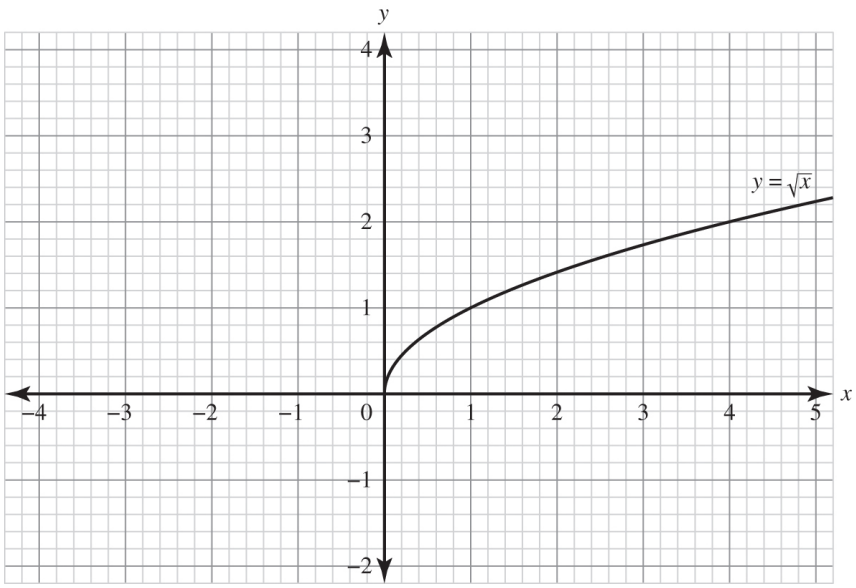


Figure 4.18

Starting with the parentheses, which in this case are under the radical, there is a horizontal shift to the left 1 unit. Then there is a vertical dilation with a scale factor of 3 and finally a reflection over the *x*-axis. The result is graphed in Figure 4.19.

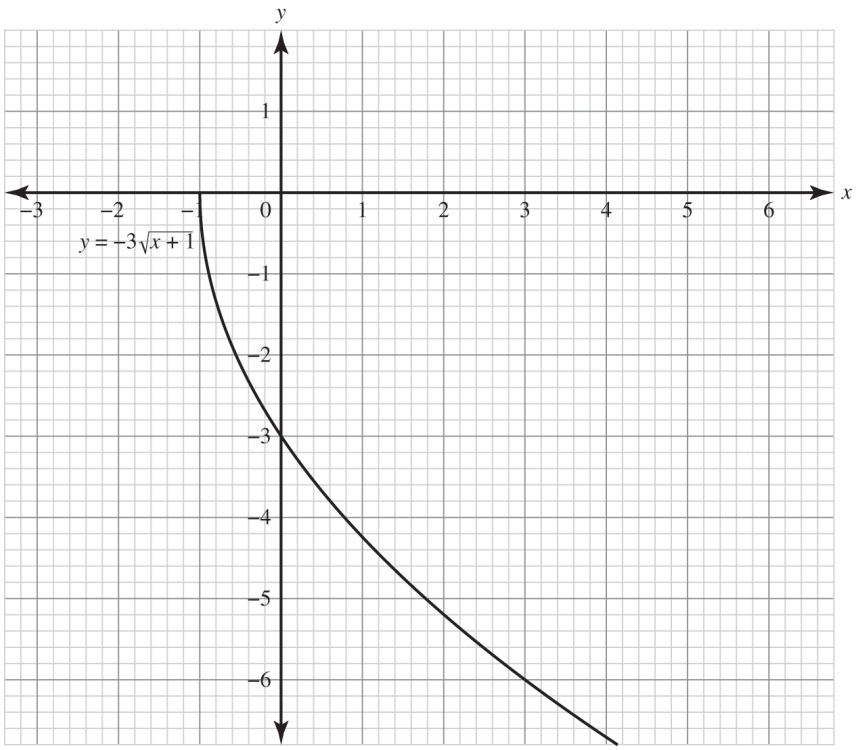


Figure 4.19

The domain of $y = -3\sqrt{x+1}$ is $x \geq -1$, and the range is $y \leq 0$.

4.3 Function Model Selection, Construction, and Application

Mathematical equations are a useful tool to model real-life scenarios that can help make predictions. It is important that an appropriate function type is used to construct a function model for a given scenario.

Table 4.3 describes the different functions discussed so far and what type of scenarios they best demonstrate.

Table 4.3

Function Type	Model Description	Example

Function Type	Model Description	Example
Linear functions	Linear functions model data sets or scenarios that demonstrate roughly constant rates of change.	Renting a car: There is a fixed cost of $150 and then a cost of $5.50 for every hour used. An equation to model this would be $y = 5.5x + 150$
Quadratic functions	Quadratic functions model data sets or scenarios that demonstrate roughly linear rates of change, or data sets that are roughly symmetric with a unique maximum or minimum value. Geometric contexts involving area or two dimensions can often be modeled using quadratic functions.	Projectile motion: A toy rocket is fired into the air from the top of a barn that is 20 yards above the ground. The height, h, of the rocket above the ground in yards after t seconds can be modeled by this function $h(t) = -5t^2 + 10t + 20$ Area: A local park has a rectangular flower bed that measures 10 feet by 15 feet. The park gardener wants to double its area by adding a strip of uniform width around the flower bed. An equation to model this would be $4x^2 + 50x + 150 = 300$
Cubic functions	Geometric contexts involving volume or three dimensions can often be modeled by cubic functions.	Volume: A box has a square base where the length of a side of the square base is x. The box is twice as tall as it is wide. An equation to model the volume of the box would be $V = (x)(x)(2x) = 2x^3$

Function Type	Model Description	Example
Polynomial functions	Polynomial functions model data sets or scenarios with multiple real zeros or multiple maxima or minima.	Temperature: A house thermostat is set to 72°F. The actual temperature in the home changes between maximum and minimum temperatures.
Piecewise functions	Piecewise functions model a data set or scenario that demonstrates different characteristics over different intervals.	Income tax: How much people earn determines their income tax rate.
Rational functions	Data sets and scenarios involving quantities that are inversely proportional can often be modeled by rational functions.	Forces: The magnitudes of both gravitational force and electromagnetic force between objects are inversely proportional to the objects' squared distance.

Linear Models

Linear functions model data sets or scenarios that demonstrate roughly constant rates of change.

Example

The data in Table 4.4 represent the price, p, and quantity demanded per day, q, of a digital camera.

Table 4.4

Price, p (in dollars)	Quantity Demanded, q
150	120
200	100
250	80
300	60

(a) Show that the quantity demanded, q, is a linear function of the price, p.

(b) State and interpret the slope.

(c) Determine the linear function that describes the relationship between p and q.

(d) State and interpret the value of the intercepts.

(e) What is the implied domain of the linear function?

✓ Solution

(a) Since each input corresponds to a single output, the quantity demanded is a function of price. Since the average rate of change is a constant—\$0.40 per camera—the function is linear.

(b) The slope is –0.4, meaning that if the price increases by \$1, the quantity demanded of digital cameras decreases by 0.4 camera.

(c) Since the slope is known, the rest of the equation can be found by substituting a known ordered pair from the table into the point-slope formula:

$$y - y_1 = m(x - x_1)$$

Each ordered pair (x, y) is represented by (p, q). Substitute the ordered pair (150, 120) into the linear equation:

$$q - 120 = -0.4(p - 150)$$

Solve for q:

$$q = -0.4p + 180$$

(d) Using the given slope-intercept form from part (c), the q-intercept is 180. This means when the price is \$0, 180 digital cameras will be demanded. To find the p-intercept, substitute 0 for q and solve for p:

$$0 = -0.4p + 180$$

$$p = \frac{-180}{-0.4} = 450$$

The p-intercept is 450, meaning that there will be 0 digital cameras demanded when the price is \$450.

(e) The implied domain is $0 \le p \le 450$.

Models can be built from linear descriptions but also from given data as shown in the previous example. When a lot of data points are given, there isn't always a constant rate of change. However, when the data points are graphed, which is called a scatterplot, it can be obvious what model best fits most data points. This is known as a regression model.

Regression models can be found using graphing utilities, such as a graphing calculator. The steps listed below show how to find regression models using the TI-84.

Steps for Finding Regression Models

1. The data must be entered into lists in the calculator. Press STAT and ENTER for 1:Edit.
2. Enter the data for the independent variable into list 1, L1. Enter the data for the dependent variable into list 2, L2. When necessary, to delete data from a list, highlight the name of the list at the top, and press the CLEAR button.
3. After entering in the data, press STAT, move to the right to select CALC, and choose the appropriate regression model for your data. Typical models would be a line of best fit, or

linear regression, which is option 4. There is also a quadratic regression (option 5).

4. After selecting the appropriate regression model, make sure the Xlist has L1 and the Ylist has L2. The FreqList and Store RegEQ can be left blank. Select Calculate at the bottom, and press ENTER.
5. The general regression model will be listed along with the values of each coefficient.

Example

A pediatrician wanted to find a model that related a child's height, *h*, to their head circumference, *c*. She randomly selects nine children from her practice, measures their height and head circumference, and obtains the data shown in Table 4.5. She plots the data on a graph, creating a scatterplot (see Figure 4.20) in which she lets *h* represent the independent variable and *c* represent the dependent variable.

Table 4.5

Height, ***h*** (inches)	Head Circumference, ***c*** (inches)
25.25	16.4
25.75	16.9
25	16.9
27.75	17.6
26.5	17.3
27	17.5
26.75	17.3

Height, ***h*** (inches)	Head Circumference, ***c*** (inches)
26.75	17.5
27.5	17.5

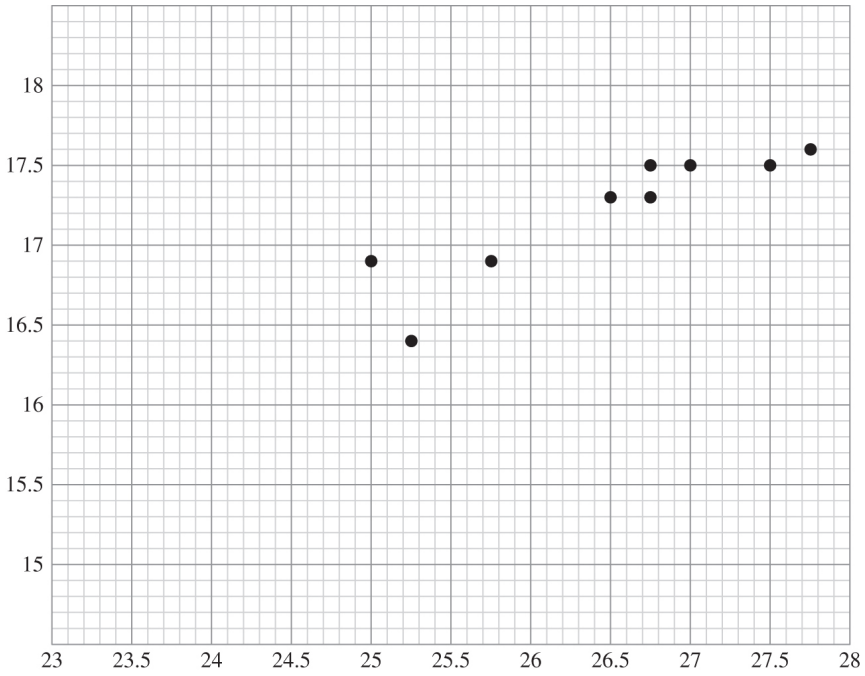


Figure 4.20

(a) Use a graphing calculator to find the line of best fit that models the relation between height and head circumference. Round all values to the nearest thousandth.

(b) Interpret the slope.

(c) Predict the head circumference of a child who is 26 inches tall.

(d) What is the height of a child whose head circumference is 17.4 inches?

☑ Solution

(a) To find the line of best fit, enter the data into L1 and L2 by pressing STAT and ENTER. After the data have been entered, press STAT and select CALC on the right. Type 4 for the linear regression model, which will provide the line of best fit. Check that L1 and L2 are listed before selecting Calculate. The calculator will display the following:

$$y = ax + b$$
$$a = .3733840304$$
$$b = 7.326806084$$

The line of best fit can be modeled by the equation $y = 0.373x + 7.327$. By substituting the variables given in the problem, this model will change to $c = 0.373h + 7.327$.

(b) The slope of the line is approximately 0.373. This means that for every inch in height, the head circumference will increase by 0.373 inches.

(c) To predict the head circumference, substitute 26 for h and solve for c:

$$c = 0.373(26) + 7.327 = 17.025$$

The head circumference is approximately 17.025 inches when the child is 26 inches tall.

(d) To predict the height, substitute 17.4 for c and solve for h.

$$17.4 = 0.373h + 7.327$$
$$10.073 = 0.373h$$
$$h = 27.00536193$$

The height of the child is approximately 27.005 inches when their head circumference is 17.4 inches.

Different strategies can be used to select the function to model a situation. One is to create a diagram. This will help identify variables, restrict domain and range if necessary, and identify the most appropriate function needed to model.

Quadratic Models

Quadratic functions model data sets or scenarios that demonstrate roughly linear rates of change, or data sets that are roughly symmetric with a unique maximum or minimum value.

Example

A rectangular piece of cardboard measuring 50 inches by 30 inches is to be made into an open box with a base of 900 in.^{2} by cutting equal squares from the four corners and then bending up the sides. Find, to the nearest tenth of an inch, the length of the side of each square that must be cut from each corner.

Solution

First create a diagram that models the given scenario. Start with a rectangle measuring 50 inches by 30 inches, as shown in Figure 4.21.

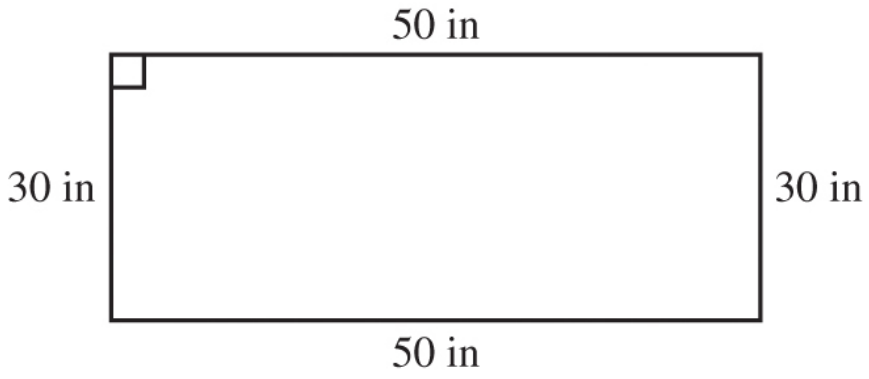


Figure 4.21

Cut out the four equal squares from each corner. Since the length of the squares is unknown, let x represent the length of the side of a square.

Since the cardboard has set dimensions and it is given that the squares are cut out so the cardboard can be bent upward, the set of numbers that x can represent needs to be restricted. See Figure 4.22. This is what is known as restricting the domain. The domain for this example is $0 < x < 15$.

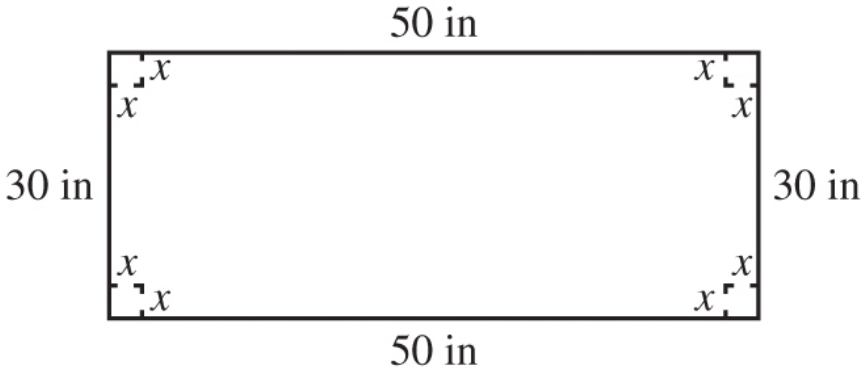


Figure 4.22

At each corner of the cardboard, the squares are being cut out. This affects each side of the cardboard since its length will be reduced by x on both ends. As shown in Figure 4.23, this changes the length and width of the cardboard.

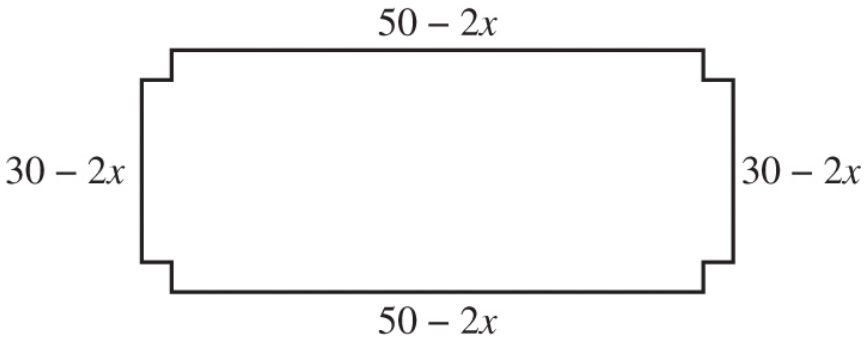


Figure 4.23

When the sides are bent up, as shown in Figure 4.24, the three-dimensional box is created and the height of the box is x.

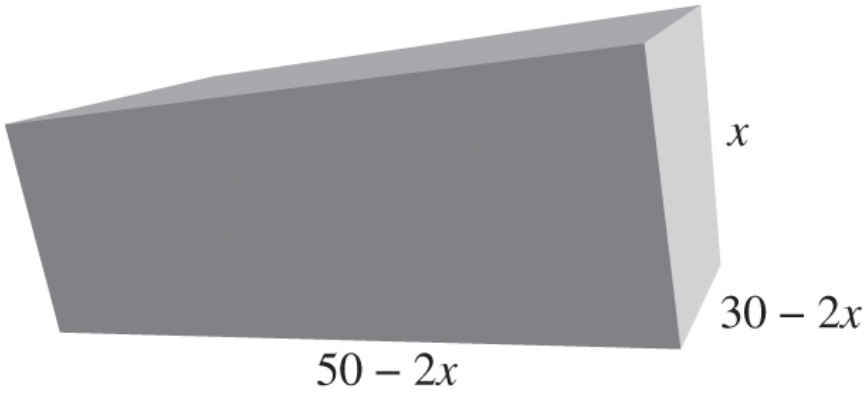


Figure 4.24

The length = 50 − 2*x*, the width = 30 − 2*x*, and the height = *x*.

The area of the rectangle is the product of its length and width. The area of the rectangle was given to be 900 in.^{2}

To solve the problem, set up the required equation:

$$900 = (50 - 2x)(30 - 2x)$$

Multiply the terms and simplify:

$$900 = 1500 - 160x + 4x^2$$

The equation can be identified as a quadratic equation. To solve for *x*, the equation must be set equal to 0. To solve for *x*, use the quadratic formula, factor, or graph to find the *x*-intercepts:

$$4x^2 - 160x + 600 = 0$$

Divide the equation by 4:

$$x^2 - 40x + 150 = 0$$

Use the quadratic formula:

$$x = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(1)(150)}}{2(1)}$$

$$x = \frac{40 \pm \sqrt{1600 - 600}}{2}$$

$$x = \frac{40 \pm \sqrt{1000}}{2}$$

$$x \approx 35.8113883, x \approx 4.188611699$$

Since the restricted domain is $0 < x < 15$, the only acceptable answer is $x \approx 4.188611699$.

The length of the side of each square is 4.2 inches.

Example

The data in Table 4.6 represent the height of the ball h of a shot putter at the instant that it has traveled x feet horizontally.

Table 4.6

Distance, x	Height, h
20	25
40	40
60	55
80	65
100	71
120	77
140	77
160	75
180	71

Distance, x	Height, h
200	64

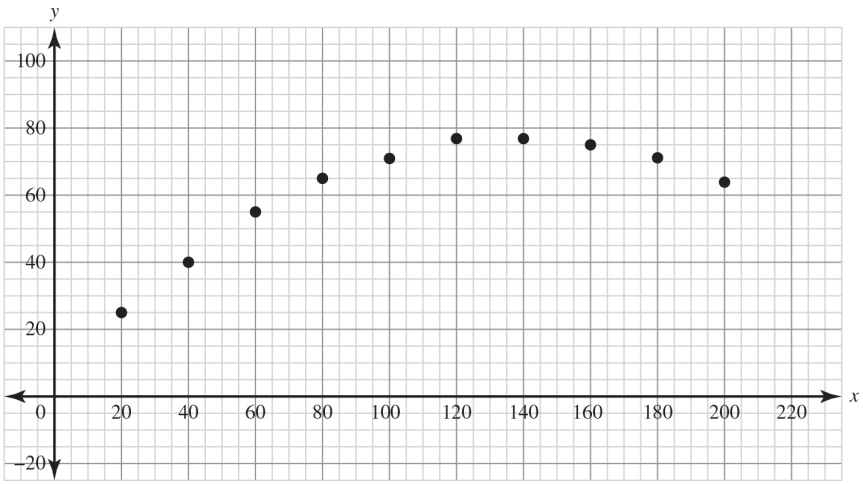


Figure 4.25

(a) Using the scatterplot in Figure 4.25, describe the type of relation that may exist between the two variables.

(b) Using your answer from part (a), find the function of best fit that models the relation between distance and height. Round all coefficients to the nearest thousandth.

(c) Use the function of best fit found in part (b) to determine how far the ball will travel before it reaches its maximum height.

(d) Use the function found in part (b) to find the maximum height of the ball.

☑ Solution

(a) Based on the scatterplot, it appears that a quadratic regression model would best fit the data.

(b) To find the quadratic regression model, enter the data into L1 and L2 by pressing STAT and ENTER. After the data have been entered, press STAT and select CALC on the right. Type 5 for the quadratic

regression model. Check that L1 and L2 are listed before selecting Calculate. The calculator will display the following:

$$y = ax^2 + bx + c$$
$$a = -0.0037121212$$
$$b = 1.031818182$$
$$c = 5.666666667$$

The quadratic regression can be modeled by the equation $y = -0.004x^2 + 1.032x + 5.667$. By substituting the variables given in the problem, this model will change to $h = -0.004x^2 + 1.032x + 5.667$.

(c) A graphing calculator can be used to find the coordinates of the maximum of the quadratic equation. Type the quadratic regression model into Y=. To see the graph better, change the window settings to the following:

Xmin = 0

Xmax = 300

Xscl = 10

Ymin = –5

Ymax = 80

Yscl = 10

Press the GRAPH button. Press 2ND TRACE and select 4:maximum. Move the cursor to the left of the maximum point and press ENTER, move the cursor to the right and press ENTER, and move the cursor close to the maximum and press ENTER for GUESS.

To the nearest integers, the coordinates of the maximum are (129, 72).

Horizontally, the ball will travel 129 feet just as it reaches its maximum height.

(d) Using the solution above, the maximum height of the ball is 72 feet.

Note that the answers in parts (c) and (d) differ from the maximum height that is provided in the table of values. This is because the

regression equation was used to calculate the maximum height. The regression equation passes through most data points, not all points. Also, the equation used had rounded coefficients, which also contributes to the values being different.

When provided with a model, it is important to closely read the problem to make sense of the context of the problem. This will help identify variables, restrict domain and range if necessary, and identify the most appropriate function needed to model.

Polynomial Models

Polynomial functions model data sets or scenarios with multiple real zeros or multiple maxima or minima.

Example

An engineer working for an automobile company is trying to help design a crash test barrier whose ideal characteristics are shown in Figure 4.26. The time after impact, t, is measured in milliseconds. The distance that the barrier has been depressed after impact, d, is measured in millimeters.

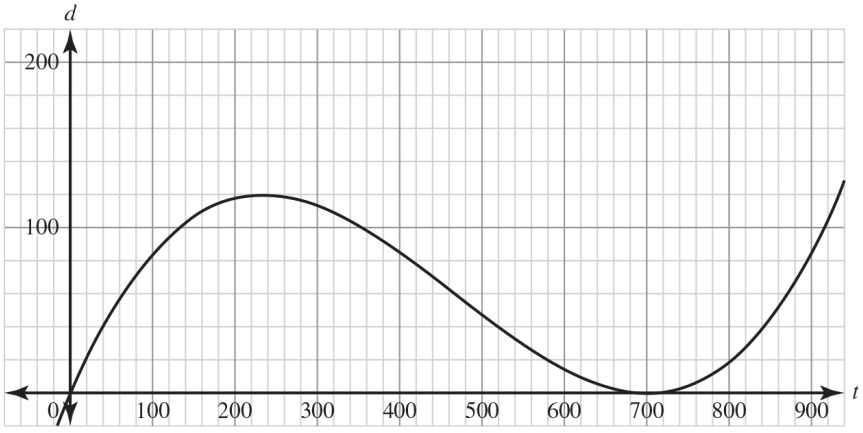


Figure 4.26

(a) If the equation for this graph is of the form $d = kt(t - a)^2$, what is the value of a? What does a represent?

(b) If the ideal crash test barrier is depressed by 100 millimeters after 130 milliseconds, find the equation of the graph given.

(b) What is the maximum amount of depression, and when does it occur? Round each to the nearest integer.

✓ Solution

(a) The given equation is a polynomial equation written in factored form. The x-intercepts will give the zeros of the graph, which yield the factors. The x-intercepts are $x = 0$ and $x = 700$. The factors of these zeros are $(x - 0) = x$ and $(x - 700)$. Since the graph is tangent at $x = 700$, the multiplicity of that zero is even, which means that $a = 700$.

When $a = 700$, this is the time that the barrier has returned to its original position.

(b) The provided information gives the ordered pair $(t, d) = (130, 100)$. Substitute these values into the equation along with $a = 700$:

$$100 = k(130)(130 - 700)^2$$

$$100 = 42237000k$$

$$k = \frac{100}{42237000} \approx 0.00000236759$$

The equation is $d = 0.00000236759t\,(t - 700)^2$.

(c) After the barrier returns to its initial position at $t = 700$, it won't continue depressing. It is important to note that this affects the range of the given graph; it won't continue increasing when $t > 700$. Therefore, the range is restricted to $0 \le y \le 120$. To find the maximum, a graphing calculator will be used to find the coordinates of the relative maximum shown in the graph.

Using a TI-84, enter in the equation by pressing the Y= button.

Type the equation into Y1. Note that d will be represented by Y1 and t will be represented by x.

To see the graph, the window will need to be changed. This can be done by pressing the WINDOW button and setting the following:

Xmin = 0

Xmax = 900

Xscl = 200

Ymin = 0

Ymax = 120

Yscl = 20

Press the GRAPH button. Press the 2ND TRACE button and select 4:maximum. Move the cursor to the left of the relative maximum, and press ENTER. Move the cursor to the right of the relative maximum, and press ENTER. Move the cursor close to the relative maximum for GUESS, and press ENTER.

The coordinates are approximately (233.333332, 119.92296).

The maximum amount of depression is 120 millimeters at 233 milliseconds.

Piecewise Models

Often scenarios don't always follow the same behavior and might change based upon different constraints such as time or money. When this occurs, the situation needs to be modeled by a piecewise function. This allows for a better fit of the situation over the different intervals.

› Example

An electric company supplies electricity to residences for a monthly customer charge of $7.58 plus 8.275 cents per kilowatt-hour (kWhr) for the first 400 kWhr supplied in the month. It then charges 6.280 cents per kWhr for all usage over 400 kWhr in the month.

(a) What is the charge for using 250 kWhr in a month?

(b) What is the charge for using 900 kWhr in a month?

(c) If C is the monthly charge for x kWhr, write a model relating the monthly charge and kilowatt-hours used.

✓ Solution

(a) To figure out the charge for using 250 kWhr, the information given for the first 400 kWhr must be used. There is the fixed monthly charge of \$7.58 and then 8.275 cents for every kWhr.

Therefore, the total charge = \$7.58 + \$0.08275(250) = \$28.27.

(b) To figure out the charge for using 900 kWhr, the cost needs to be broken into two parts: the first 400 kWhr and the usage over 400 kWhr. There is the fixed monthly charge of \$7.58, then 8.275 cents for every kWhr for the first 400 kWhr, and then the charge changes to 6.280 cents for the remaining 500 kWhr.

Therefore, the total charge = \$7.58 + \$0.08275(400) + \$0.06280(500) = \$72.08.

(c) The steps to finding the solution in part (b) can help write a general equation that models the given scenario. It is important to first identify the independent variable. Let x = the number of kilowatt-hours used. If $0 \le x \le 400$, the monthly charge, C, in dollars can be found by adding the monthly charge \$7.58 to the product of x and \$0.08275. So if $0 \le x \le 400$, $C(x) = \$7.58 + \$0.08275x$.

If $x > 400$, the increased kWhr charge of \$0.06280 is multiplied by the amount of kWhr that is more than 400. This is represented by $x - 400$. So if $x > 400$, $C(x) = \$7.58 + \$0.08275x + \$0.06280(x - 400)$.

Written as a piecewise function:

$$C(x) = \begin{cases} \$7.58 + \$0.08275x, & 0 \le x \le 400 \\ \$7.58 + \$0.08275x + \$0.06280(x - 400), & x > 400 \end{cases}$$

Rational Models

Data sets and aspects of contextual scenarios involving quantities that are inversely proportional can often be modeled by a rational function.

Inverse variation is the relationship between variables that are represented in the form $y = \frac{k}{x}$, where x and y are two variables and k is the constant value or constant of proportionality. It states that if the value of one quantity increases, the value of the other quantity decreases. Inverse variation means that a variable is inversely varying with respect to another variable. It represents the inverse relationship between two quantities. Hence, a variable is inversely proportional to another variable.

There are many real-life examples of inverse variation. One example is if the distance traveled by a train at a constant speed increases, the time remaining for the ride decreases. A second example is if the number of people added to a job increases, the time taken to accomplish the job decreases.

› Example

If x varies inversely with y and if $x = 10$ and $y = 4$, then what is the value of the constant of variation?

✓ Solution

Let k represent the constant of variation. Since x varies inversely with y, $x = \frac{k}{y}$ or $xy = k$. Substitute the given values for x and y:

$$10 = \frac{k}{4}$$

$$k = 40$$

The constant of variation is 40.

› Example

If 24 workers can build a house in 40 days, how many workers will be required to build the same house in 20 days?

✓ Solution

Let x^2 represent the number of workers employed to build the house in 20 days. The time taken to build the house is inversely proportional to the

number of workers required. So the following equation can be used:

$$x_1 y_1 = x_2 y_2$$

Substitute the known information:

$$(24)(40) = (x_2)(20)$$

$$x_2 = 48$$

It will take 48 workers to build the house in 20 days.

Rational functions can also appear in geometric models as well.

Example

A rectangular area adjacent to a river is to be fenced in; no fence is needed on the river side. The enclosed area is to be 1,000 square feet. Fencing for the side parallel to the river is $5 per foot, and fencing for the other two sides is $8 per foot. There are four corner posts that are $25 each. Let x be the length of one of the sides perpendicular to the river.

(a) Write a function $C(x)$ that describes the cost of the project.

(b) State the domain of $C(x)$.

(b) Find the dimensions of the least expensive enclosure.

Solution

First create a diagram that models the given scenario. Start with a rectangle with an area of 1,000 square feet, as shown in Figure 4.27. Label one side as the river and the two perpendicular sides to the river as x.

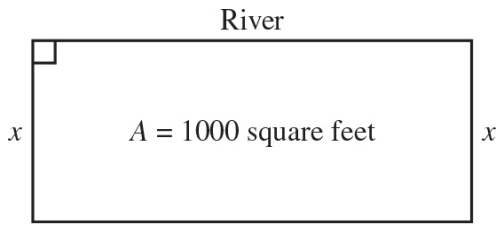


Figure 4.27

The length of the side opposite the river is also unknown. Instead of assigning it a new variable, use the given area information to represent its length in terms of x:

$$\text{Area of a rectangle} = (\text{length})(\text{width})$$

$$1000 = (\text{length})(x)$$

$$\text{length} = \frac{1000}{x}$$

Figure 4.28 shows the diagram with all four sides labeled.

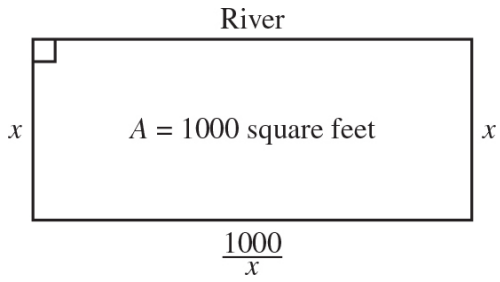


Figure 4.28

Since the rectangle has a set area, the set of numbers that x can represent needs to be restricted. The domain for this example is $x > 0$.
Now that the sides of the model are labeled, the cost of the project can be determined by multiplying each side by its respective cost per foot and adding on any additional charges.

(a) $C(x) = 5\left(\frac{1000}{x}\right) + 8(x) + 8(x) + 25(4) = 100 + 16x + \frac{5000}{x}$

(b) The domain is $x > 0$.

(c) To find the dimensions of the least expensive enclosure, graph the function, as shown in Figure 4.29, and use the calculator to find the coordinates of the minimum point.

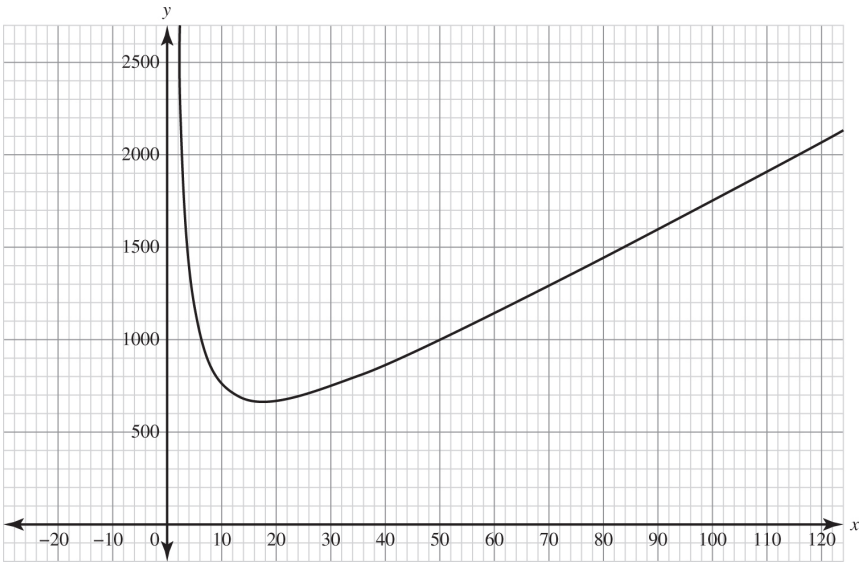


Figure 4.29

A good viewing window would be:

Xmin = 0

Xmax = 120

Xscl = 10

Ymin = –5

Ymax = 2500

Yscl = 100

Press the GRAPH button. To find the minimum point, press 2ND TRACE and select 3:minimum. Move the cursor to the left and right of the minimum point to obtain the coordinates (17.67761, 665.68542).

The dimensions to yield the least expensive enclosure would have a width of approximately 17.7 feet and a length of approximately $\frac{1000}{17.67761} = 56.6$ feet.

4.4 Composition of Functions

Two functions may be linked to form a new function by using the output of one function as the input to the other function. The new function that is

created is called a composite function.

Given two functions f and g, the composite function, denoted by $f \circ g(x)$, which is read as "f composed with g" or "f of g of x," is defined by:

$$f \circ g(x) = f(g(x))$$

where the domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

In other words, if f and g are functions, the composite function $f(g(x))$ maps a set of input values to a set of output values such that the output values of g are used as input values of f.

There are many properties associated with composite functions. To help derive these properties, the following composite functions will be evaluated.

› Example

Let $f(x) = x - 2$ and $g(x) = 10x$. Find each of the following.

1. $f \circ g(1)$
2. $g \circ f(1)$
2. $f \circ f(-2)$
2. $g \circ g(-1)$

✓ Solution

For functions f and g, if the output of function g is used as the input to function f, the composite function is denoted as $f \circ g(x) = f(g(x))$. This composition allows you to consider g the "inside" function and shows where the problem should begin to be evaluated. When evaluating composite functions, first evaluate the inside function and work your way to the outside functions.

Order of Composition

The composition of functions is not commutative. The values of $f(g(x))$ and $g(f(x))$ are usually not equal for the same x. The order in which functions are composed matters.

1. $f \circ g(1) = f(g(1))$. First evaluate $g(1) = 10(1) = 10$. Then $f(g(1)) = f(10) = (10) - 2 = 8$.
2. $g \circ f(1) = g(f(1))$. First evaluate $f(1) = (1) - 2 = -1$. Then $g(f(1)) = g(-1) = 10(-1) = -10$.
3. $f \circ f(-2) = f(f(-2))$. First evaluate $f(-2) = (-2) - 2 = -4$. Then $f(f(-2)) = f(-4) = (-4) - 2 = -6$.
4. $g \circ g(-1) = g(g(-1))$. First evaluate $g(-1) = 10(-1) = -10$. Then $g \circ g(-1) = g(-10) = 10(-10) = -100$.

When considering the solutions from (1) and (2), it can be observed that $f \circ g(1) \neq g \circ f(1)$. Therefore, the composition of functions is not commutative.

› Example

Let $f(x) = 2x - 1$, $f(x) = 2x - 1, g(x) = \sqrt{x+7}$, and $h(x) = x$. Find each of the following.

1. $f(g(x))$
2. $g(f(x))$
2. $f(h(x))$
2. $h(f(x))$
2. $f(g(x + 2))$

✓ Solution

1. The inside function is $g(x)$, which equals $\sqrt{x+7}$. That becomes the input for the outside function, f. Substitute $\sqrt{x+7}$ for every x-term in $f(x)$:

$$f(g(x)) = f(\sqrt{x+7}) = 2(\sqrt{x+7}) - 1 = 2\sqrt{x+7} - 1$$

2. The inside function is $f(x)$, which equals $2x - 1$. That becomes the input for the outside function, g. Substitute $2x - 1$ for every x-term in $g(x)$:

$$g(f(x)) = g(2x - 1) = \sqrt{(2x - 1) + 7} = \sqrt{2x - 1 + 7} = \sqrt{2x + 6}$$

3. The inside function is $h(x)$, which equals x. That becomes the input for the outside function, f. Substitute x for every x-term in $f(x)$:

$$f(h(x)) = f(x) = 2(x) - 1 = 2x - 1$$

4. The inside function is $f(x)$, which equals $2x - 1$. That becomes the input for the outside function, h. Substitute $2x - 1$ for every x-term in $h(x)$:

$$h(f(x)) = h(2x - 1) = (2x - 1) = 2x - 1$$

5. The inside function is $g(x)$. However, the input for $g(x)$ is something other than x. First evaluate $g(x + 2)$ by substituting $x + 2$ in for every x in $g(x)$:

$$g(x + 2) = \sqrt{(x + 2) + 7} = \sqrt{x + 2 + 7} = \sqrt{x + 9}$$

That output becomes the input for the outside function, f. Substitute $\sqrt{x + 9}$ for every x-term in $f(x)$:

$$f(g(x + 2)) = f(\sqrt{x + 9}) = 2(\sqrt{x + 9}) - 1 = 2\sqrt{x + 9} - 1$$

Like in the previous example, when considering the solutions from (1) and (2), it can be observed that $f(g(x)) \neq g(f(x))$. It has been proved in general that the composition of functions is not commutative; that is, $f(g(x))$ and $g(f(x))$ are typically different functions.

When considering the solutions from (3) and (4), it can be observed that $f(h(x)) = f(x)$ and $h(f(x)) = f(x)$. This might appear to contradict the noncommutative property of composite functions. However, this is a special case.

A Special Case for Compositions

If the function $f(x) = x$ is composed with any function g, the resulting composite function is the same as g; that is, $g(f(x)) = f(g(x)) = g(x)$.

The function $f(x) = x$ is called the identity function.

Values for the composite function $f(g(x))$ can be calculated or estimated for the algebraic, graphical, numerical, or verbal representations of f and g by using output values from g as input values for f.

Example

Evaluate each expression using the values given in Table 4.7.

Table 4.7

x	–3	–2	–1	0	1	2	3
$f(x)$	–7	–5	–3	–1	3	5	5
$g(x)$	8	3	0	–1	0	3	8

1. $f(g(2))$
2. $g(f(0))$
2. $f \circ g(0)$
2. $g \circ g(-2)$

Solution

1. To evaluate $f(g(2))$, first evaluate $g(2)$ by locating $x = 2$ on the table and sliding down to the row of $g(x)$: $g(2) = 3$. Then evaluate $f(3)$ by locating $x = 3$ and sliding down to the row of $f(x)$:

$$f(g(2)) = f(3) = 5$$

2. To evaluate $g(f(0))$, first evaluate $f(0)$ by locating $x = 0$ on the table and sliding down to the row of $f(x)$: $f(0) = -1$. Then evaluate $g(-1)$ by locating $x = -1$ and sliding down to the row of $g(x)$:

$$g(f(0)) = g(-1) = 0$$

3. To evaluate $f \circ g(0)$, first evaluate $g(0)$ by locating $x = 0$ on the table and sliding down to the row of $g(x)$: $g(0) = -1$. Then evaluate $f(-1)$ by locating $x = -1$ and sliding down to the row of $f(x)$:

$$f \circ g(0) = f(g(0)) = f(-1) = -3$$

4. To evaluate $g \circ g(-2)$, first evaluate $g(-2)$ by locating $x = -2$ on the table and sliding down to the row of $g(x)$: $g(-2) = 3$. Then evaluate $g(3)$ by locating $x = 3$ and sliding down to the row of $g(x)$:

$$g \circ g(-2) = g(g(-2)) = g(3) = 8$$

Example

Evaluate each expression using the values given in the graphs of $f(x)$ and $g(x)$ shown in Figure 4.30.

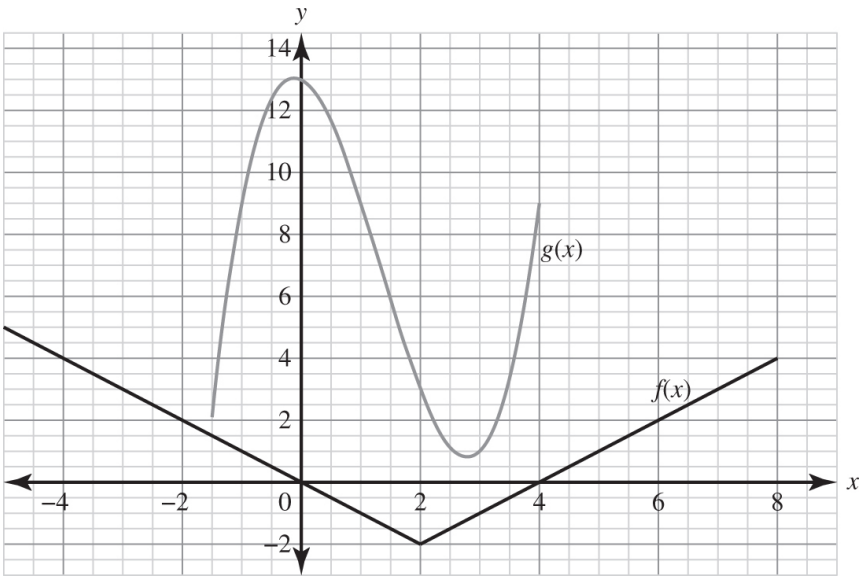


Figure 4.30

1. $f(g(2))$
2. $g(f\,(0))$
2. $f(f\,(6))$
2. $g(g(-\,1.5))$

✓ Solution

1. To evaluate $f(g(2))$, first evaluate $g(2)$ by locating $x = 2$ on the x-axis and sliding up to the graph of $g(x)$ to find the y-coordinate of the point on the graph: $g(2) = 3$. Then evaluate $f\,(3)$ by locating $x = 3$ on the x-axis and sliding down to the graph of $f(x)$ to find the y-coordinate of the point on the graph:

$$f(g(2)) = f(3) = -1$$

2. To evaluate $g(f\,(0))$, first evaluate $f\,(0)$ by locating $x = 0$ on the x-axis and locating the point on the graph of $f(x)$ to find the y-coordinate: $f\,(0) = 0$. Then evaluate $g(0)$ by locating $x = 0$ on the x-axis and sliding up to the graph of $g(x)$ to find the y-coordinate of the point on the graph:

$$g(f(0)) = g(0) = 13$$

3. To evaluate $f(f\,(6))$, first evaluate $f\,(6)$ by locating $x = 6$ on the x-axis and sliding up to the graph of $f(x)$ to find the y-coordinate of the point on the graph: $f\,(6) = 2$. Then evaluate $f\,(2)$ by locating $x = 2$ on the x-axis and sliding down to the graph of $f(x)$ to find the y-coordinate of the point on the graph:

$$f(f(6)) = f(2) = -2$$

4. To evaluate $g(g(-\,1.5))$, first evaluate $g(–1.5)$ by locating $x = –1.5$ on the x-axis and sliding up to the graph of $g(x)$ to find the y-coordinate of the point on the graph: $g(-\,1.5) = 2$. Then evaluate $g(2)$ by locating $x = 2$ on the x-axis and sliding up to the graph of $g(x)$ to find the y-coordinate of the point on the graph:

$$g(g(-1.5)) = g(2) = 3$$

Domain of a Composite Function

The domain of the composite function $f(g(x))$ is restricted to those input values of g for which the corresponding output values are in the domain of f. The composite function $f(g(x))$ uniquely maps input values of g to output values of f, dependent on the domain restrictions of f and g.

› Example

Let $f(x) = x^2 - 3x + 2$ and $g(x) = -4x + 7$. Find:

(a) $f \circ g(x)$

(b) The domain of $f \circ g(x)$

✓ Solution

(a) The inside function is $g(x)$, which equals $-4x + 7$. That becomes the input for the outside function, f. Substitute $-4x + 7$ for every x-term in $f(x)$:

$$\begin{aligned} f(g(x)) &= f(-4x+7) = (-4x+7)^2 - 3(-4x+7) + 2 \\ &= 16x^2 - 28x - 28x + 49 + 12x - 21 + 2 \\ f(g(x)) &= 16x^2 - 44x + 30 \end{aligned}$$

(b) Since the domains of both f and g are the set of real numbers, the domain of $f \circ g(x)$ is the set of real numbers.

› Example

Find the domain of $f \circ g(x)$ if $f(x) = \frac{1}{x-3}$ and $g(x) = \frac{-3}{x+7}$.

✓ Solution

Since $f \circ g(x) = f(g(x))$, first consider the domain of $g(x)$. The domain of g is all real numbers, $x \neq -7$. So $x = -7$ is excluded from the domain of $f \circ g(x)$. The domain of f is all real numbers, $x \neq 3$. So $g(x)$ cannot equal 3. It

needs to be determined what x-values are the solution to $g(x) = 3$ so those x-values can be excluded from the domain:

$$\begin{aligned} g(x) &= 3 \\ \frac{-3}{x+7} &= 3 \\ -3 &= 3(x+7) \\ -1 &= x+7 \\ x &= -8 \end{aligned}$$

The domain of $f \circ g(x)$ is all real numbers, $x \neq -8$, $x \neq -7$.

Some techniques in calculus make it necessary to be able to determine the different functions of a composite function. For example, the function $c(x) = |x^2 - 3x|$ is the composition of the functions f and g, where $f(x) = |x|$ and $g(x) = x^2 - 3x$, because $c(x) = f(g(x)) = f(x^2 - 3x) = |x^2 - 3x|$.

Example

Find functions $f(x)$ and $g(x)$ such that $f(g(x)) = c(x)$ and $c(x) = \sqrt{3x - 7}$.

Solution

The outer function is the square root function, and the inner function is the radicand $3x - 7$. Let $g(x) = 3x - 7$ and $f(x) = \sqrt{x}$. Then $f(g(x)) = f(3x - 7) = \sqrt{3x - 7} = c(x)$.

Example

Find functions $f(x)$ and $g(x)$ such that $f(g(x)) = c(x)$ and $c(x) = 2 \cdot \frac{1}{x+5} - 3$.

Solution

The inner function is the rational function $\frac{1}{x+5}$. If that expression is replaced by an x, it reveals that the outer function is $2x - 3$. Let $g(x) = \frac{1}{x+5}$ and $f(x) = 2x - 3$. Then $f(g(x)) = f\left(\frac{1}{x+5}\right) = 2 \cdot \frac{1}{x+5} - 3 = c(x)$.

4.5 Inverse Functions

For a relation to be a function, every element in its domain must have one and only one corresponding element in its range. On a specified domain, a function, f, has an inverse function or is invertible if each output value of f is mapped from a unique input value. The domain of a function may be restricted to make the function invertible.

An inverse function can be thought of as a reverse mapping of the function. An inverse function, f^{-1}, maps the output values of a function, f, on its invertible domain to their corresponding input values.

Inverse Functions and Input-Output Pairs

Given function $f(x)$, if $f(a) = b$, then $f^{-1}(b) = a$.

If a function consists of input-output pairs (a, b), the inverse function consists of input-output pairs (b, a).

› Example

(a) Find the inverse of the following function:

$$\{(-3, -27), (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8), (3, 27)\}$$

(b) State the domain and range of the function and its inverse.

(b) Determine if the inverse is also a function.

✓ Solution

(a) The inverse of the given function is found by interchanging the entries in each ordered pair:

$$\{(-27, -3), (-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$$

(b) The domain of the given function is

$$\{-3, -2, -1, 0, 1, 2, 3\}$$

The range of the given function is

$$\{-27, -8, -1, 0, 1, 8, 27\}$$

The domain of the inverse is

$$\{-27, -8, -1, 0, 1, 8, 27\}$$

The range of the inverse is

$$\{-3, -2, -1, 0, 1, 2, 3\}$$

(c) The inverse is a function since every element in the domain maps to one and only one element in the range.

› Example

(a) Find the inverse of the following function:

$$\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$$

(b) State the domain and range of the function and its inverse.

(b) Determine if the inverse is also a function.

✓ Solution

(a) The inverse of the given function is found by interchanging the entries in each ordered pair:

$$\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$$

(b) The domain of the given function is

$$\{-3, -2, -1, 0, 1, 2, 3\}$$

The range of the given function is

$$\{0, 1, 4, 9\}$$

The domain of the inverse is

$$\{0, 1, 4, 9\}$$

The range of the inverse is

$$\{-3, -2, -1, 0, 1, 2, 3\}$$

(c) The inverse is not a function since $x = 1$, $x = 4$, and $x = 9$ map to more than one element in the range.

A one-to-one function, f, has an inverse, f^{-1}, that is also a function.

› Example

Determine whether the following functions are one-to-one.

1. $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$
2. $\{(-2, 6), (-1, 3), (0, 2), (1, 5), (2, 8)\}$

A function is a one-to-one function if any two different inputs in the domain correspond to two different outputs in the range.

If x_1 and x_2 are two different inputs of a function f, then f is one-to-one if $f(x_1) \neq f(x_2)$.

A function is not one-to-one if two different inputs correspond to the same output.

✓ Solution

1. The function is not one-to-one because there are three different pairs of inputs that correspond to the same output: –3 and 3 correspond to 9, –2 and 2 correspond to 4, and –1 and 1 correspond to 1.
2. The function is one-to-one because there are no two distinct inputs that correspond to the same output.

For functions defined by an equation $y = f(x)$ and for which the graph is known, the horizontal line test can be performed to determine whether a function is one-to-one.

The Horizontal Line Test

If every horizontal line intersects the graph of a function f in at most one point, f is one-to-one.

Example

For each function, use its graph to determine whether the function is one-to-one.

1. $f(x) = x^4$
2. $g(x) = x^5$

Solution

1. As shown in Figure 4.31, a horizontal line intersects the graph twice. So $f(x)$ is not one-to-one.

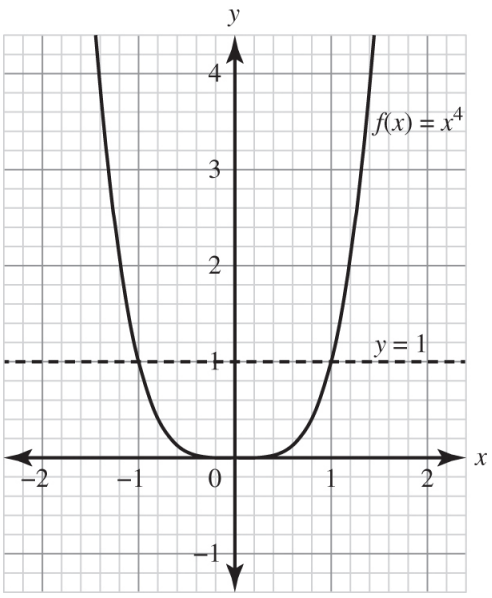


Figure 4.31

2. As shown in Figure 4.32, every horizontal line intersects the graph exactly once. So $g(x)$ is one-to-one.

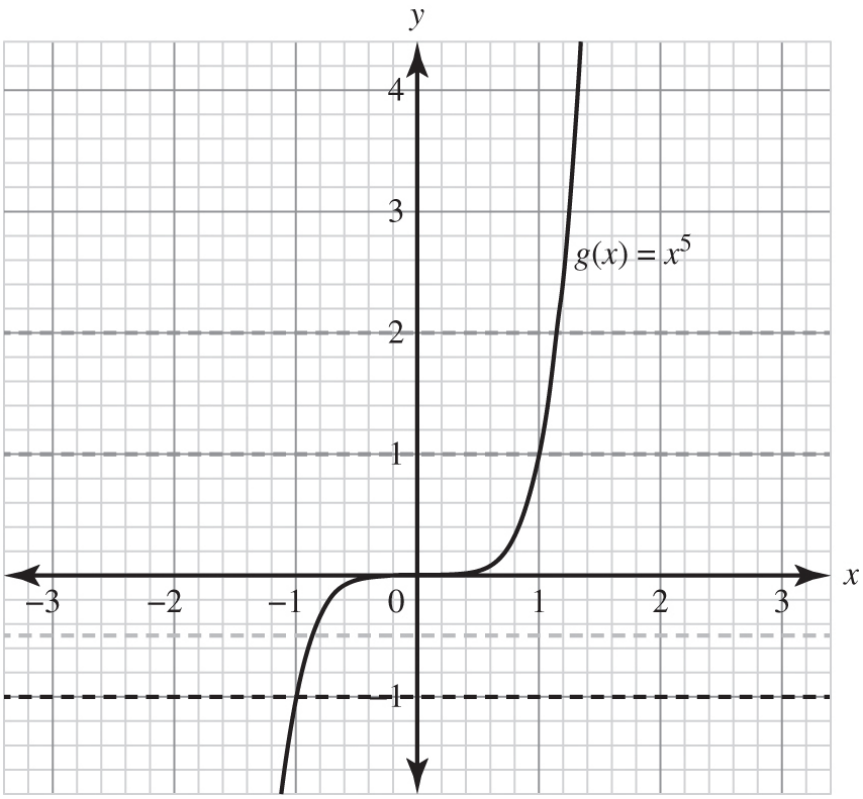


Figure 4.32

Another interesting graphical relationship between a function and its inverse deals with symmetry. The inverse of the graph of the function $y = f(x)$ can be found by reversing the roles of the x- and y-axes, that is, by reflecting the graph of the function over the line $y = x$.

Example

Given the graph of the function $f(x)$ in Figure 4.33, use the set of ordered pairs to graph its inverse function, $f^{-1}(x)$, on the same set of axes.

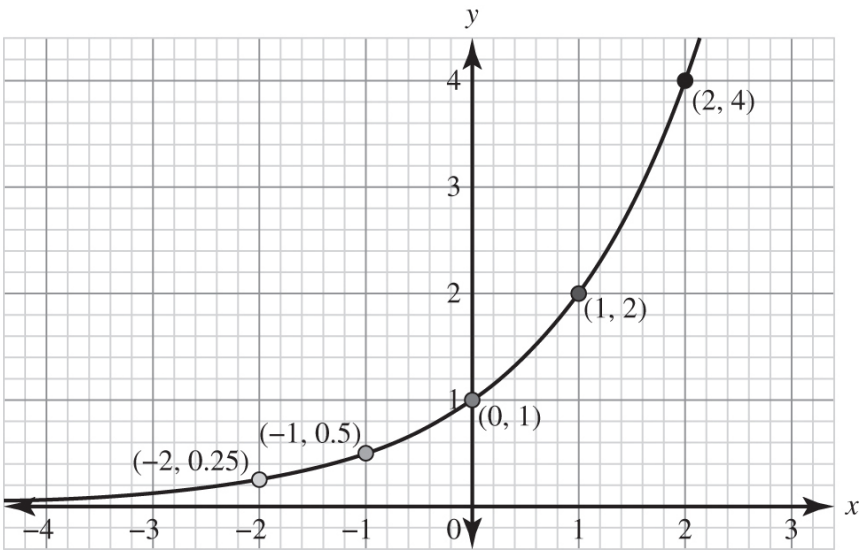


Figure 4.33

☑ Solution

Since this graph passes the horizontal line test, the function has an inverse that is also a function.

Reversing the input-output pairs of the function gives a set of ordered pairs for the inverse function. The following is a set of points on the graph of the inverse function:

$$\{(0.25, -2), (0.5, -1), (1, 0), (2, 1), (4, 2)\}$$

Use this set of ordered pairs to graph the inverse function, as shown in Figure 4.34:

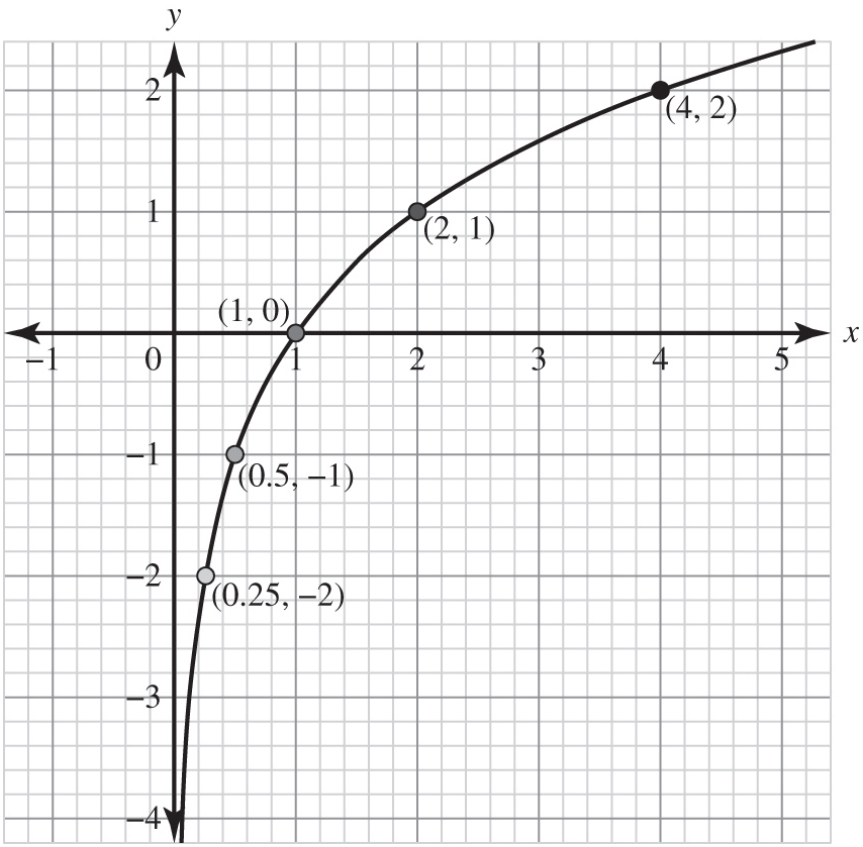


Figure 4.34

As a check, graph the original function and its inverse along with the line $y = x$ to check for symmetry. This is shown in Figure 4.35.

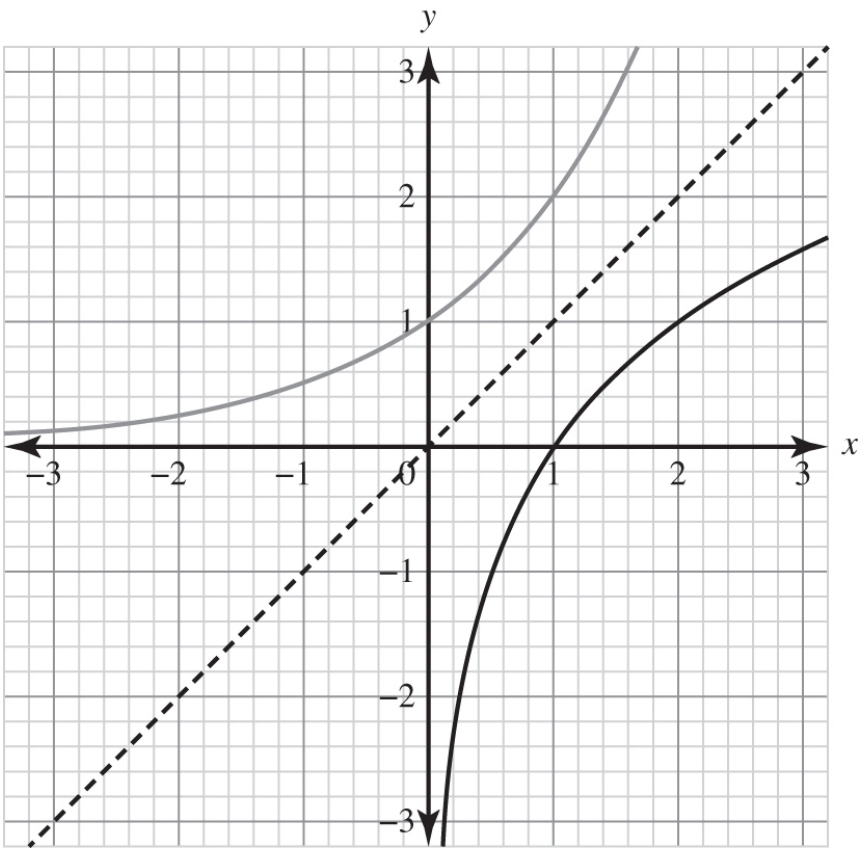


Figure 4.35

There are also algebraic techniques to determine the inverse of a function.

The Identity Function

The composition of a function, f, and its inverse function, f^{-1}, is the identity function. In other words, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Example

State whether the following functions are inverses of each other.

1. $f(x) = 3x - 4$ and $g(x) = \frac{x}{3} + \frac{4}{3}$

2. $h(x) = x^2 - 4$ and $i(x) = \sqrt{x^2 + 4}$

✓ Solution

One way to verify if two equations are inverses of each other is to take their composition in both directions and show that both are equal to x.

1.

$$f(g(x)) = f\left(\frac{x}{3} + \frac{4}{3}\right) = 3\left(\frac{x}{3} + \frac{4}{3}\right) - 4 = x + 4 - 4 = x$$

$$g(f(x)) = g(3x - 4) = \frac{3x - 4}{3} + \frac{4}{3} = \frac{3x - 4 + 4}{3} = \frac{3x}{3} = x$$

Since $f(g(x)) = g(f(x)) = x$, the two functions are inverses of each other.

2.

$$h(i(x)) = h\left(\sqrt{x^2 + 4}\right) = \left(\sqrt{x^2 + 4}\right)^2 - 4 = x^2 + 4 - 4 = x^2$$

$$i(h(x)) = i(x^2 - 4) = \sqrt{(x^2 - 4)^2 + 4} = \sqrt{x^4 - 8x^2 + 16 + 4} = \sqrt{x^4 - 8x^2 + 20}$$

Since $h(i(x)) \neq i(h(x))$ and they both did not equal x, they are not inverses of each other.

It is possible to find the equation of a function's inverse algebraically. The inverse of the function can be found by determining the inverse operations to reverse the mapping.

Algebraically Finding the Inverse

One method for finding the inverse of the function f is reversing the roles of x and y in the equation $y = f(x)$ and then solving for $y = f^{-1}(x)$.

› Example

Find the inverse equation of $f(x) = 3x + 7$. Verify that the original equation and its inverse equation are both functions.

✓ Solution

The given equation graphs a line with a y-intercept at (0, 7) and has a slope of 3. This graph would pass the horizontal line test, and so both the given equation and its inverse will be functions.

To find the inverse, reverse the x and the y:

$$x = 3y + 7$$

That equation represents the inverse, but it is more common to solve for y:

$$x - 7 = 3y$$

$$y = \frac{x - 7}{3}$$

The inverse function is $f^{-1}(x) = \frac{x - 7}{3}$.

› Example

Given the function $f(x) = \frac{2x - 1}{x + 1}, x \neq -1$:

(a) State whether the function is one-to-one.

(b) Find the equation of its inverse.

✓ Solution

(a) To find whether the function is one-to-one, graph $f(x)$, as shown in Figure 4.36, to see if it passes the horizontal line test.

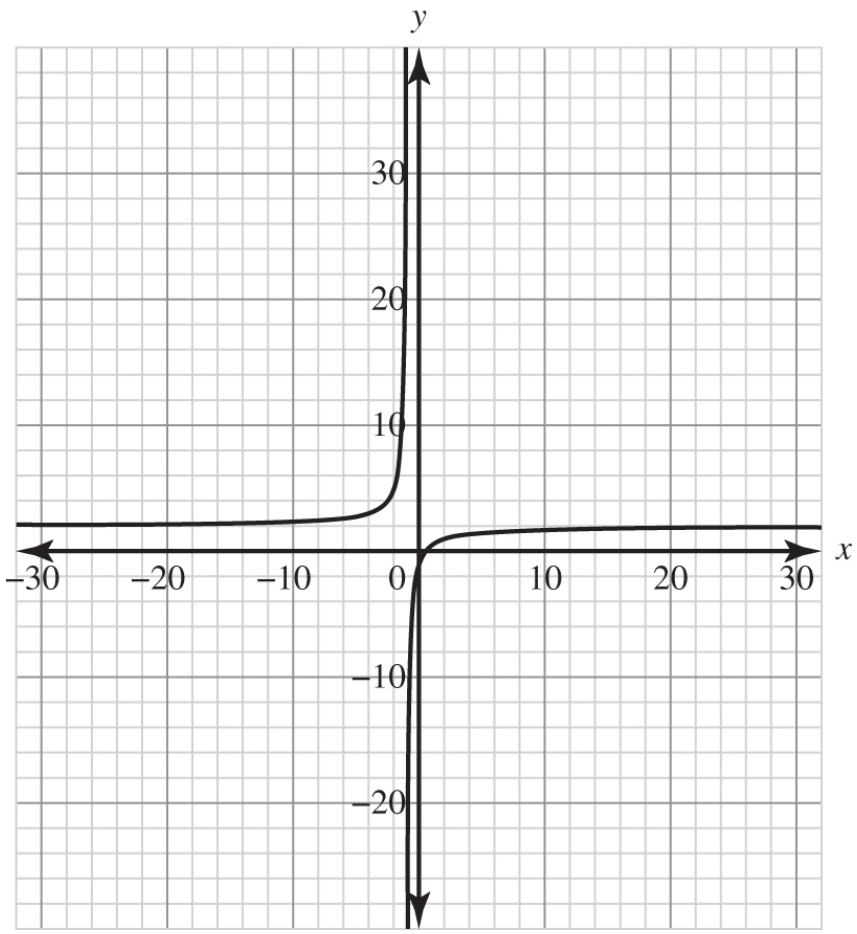


Figure 4.36

Any horizontal line drawn will intersect the graph once and only once. Therefore, the function is one-to-one and its inverse will also be a function.

(b) To find the inverse function, reverse the x and the y:

$$x = \frac{2y - 1}{y + 1}$$

That equation represents the inverse, but it is more common to solve for y:

$$
\begin{aligned}
x(y+1) &= 2y-1 \\
xy+x &= 2y-1 \\
xy-2y &= -1-x \\
y(x-2) &= -1-x \\
y &= \frac{-1-x}{x-2}
\end{aligned}
$$

The inverse function is $f^{-1}(x) = \frac{-1-x}{x-2}$.

Not all functions are one-to-one. So to have an inverse that is also a function, it is necessary to restrict the domain of a function. Here the horizontal line test can be a useful visual aid to see where to limit the domain of a function so that the horizontal lines will intersect the function only once.

Example

Find the inverse function of $f(x) = x^2$.

Solution

While $f(x) = x^2$ is a function, it is not a one-to-one function as it fails the horizontal line test. This is shown in Figure 4.37.

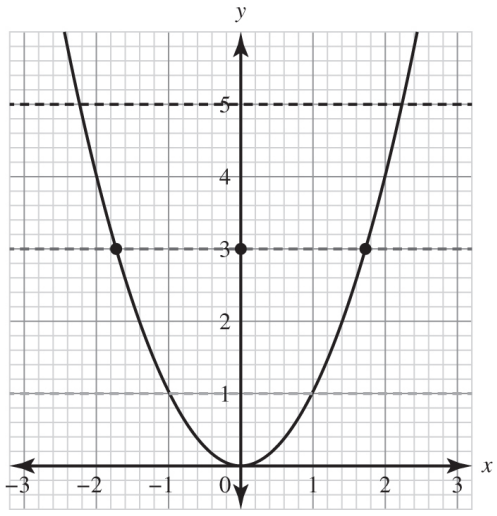


Figure 4.37

Using this visual, if the domain is restricted to $x \geq 0$, the horizontal lines intersect the graph only once, as shown in Figure 4.38.

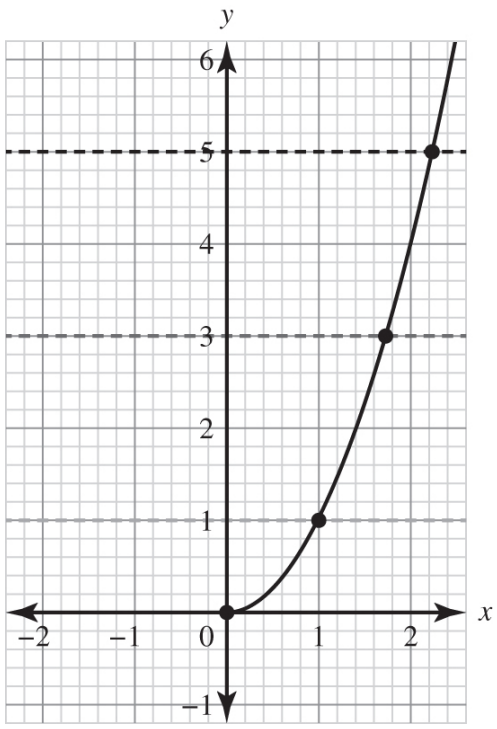


Figure 4.38

The function $f(x) = x^2$, $x \geq 0$ will have an inverse function.

To find the equation of the inverse function, reverse the x and the y:

$$x = y^2, y \geq 0$$

Solve for y by extracting the square root:

$$y = \pm\sqrt{x}$$

Since $y \geq 0$, there is only one solution:

$$y = \sqrt{x}$$

Therefore, the inverse function is $f^{-1}(x) = \sqrt{x}$.

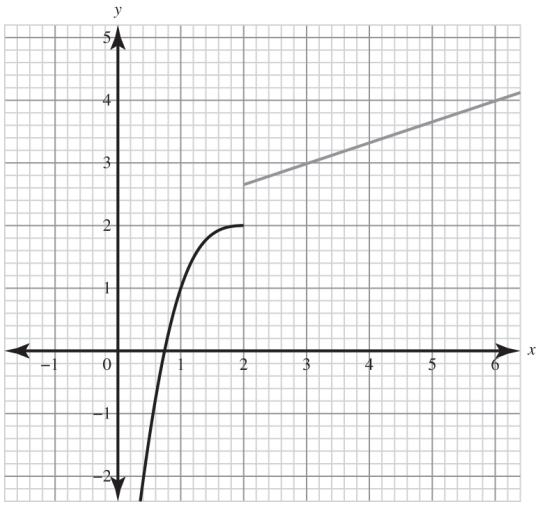
Multiple-Choice Questions

1. If $f(x) = x^2 + 1$, then $f(a - 1) =$

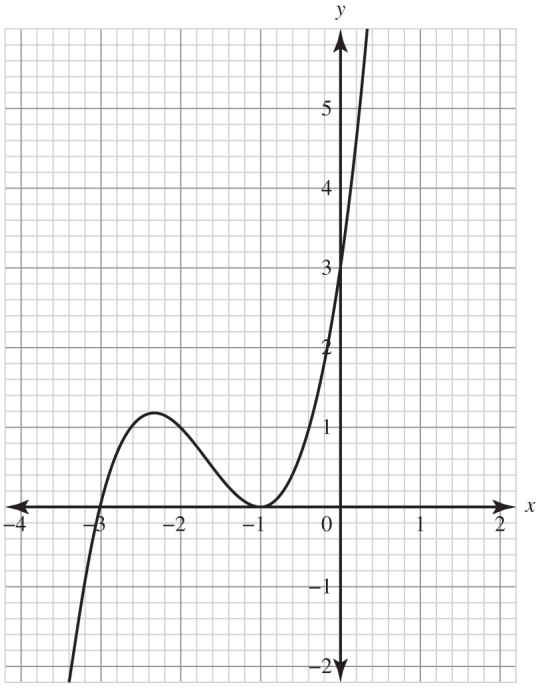
   1. (A) $a^2 + 1$
   2. (B) a^2
   3. (C) $a^2 - 1$
   4. (D) $a^2 - 2a + 2$

2. Which of the accompanying diagrams depicts a one-to-one function?

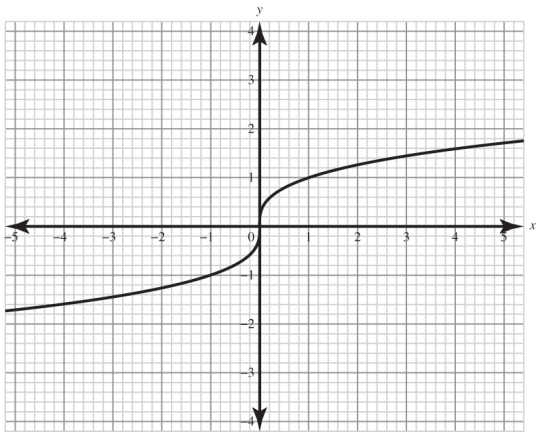
(A)



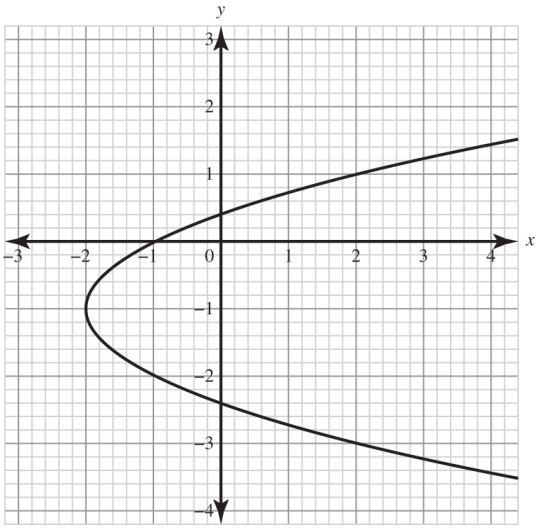
(B)



(C)



(D)



3. If $f(x) = x^2 + 1$, then $\frac{f(x+h) - f(x)}{h} =$

1. (A) $2x + h$
2. (B) $2x + h + 2$
3. (C) h
4. (D) $\frac{h^2 + 2}{h}$

4. A ball is thrown into the air such that its height, h, in feet at any time, t, in seconds is given by the function $h(t) = -16t^2 + 80t + 10$. What is the maximum height, in feet, attained by the ball?

1. (A) 140
2. (B) 110
3. (C) 85
4. (D) 10

5. If the function f is defined by $f(x) = 3x + k$ and the function g is defined by $g(x) = \frac{x-1}{3}$, for what value of k is $f(g(x)) = g(f(x))$?

1. (A) -1
2. (B) 0
3. (C) 1
4. (D) k does not exist

6. What is the equation of the function $h(x)$ whose graph can be obtained by transforming the graph of $f(x) = 2\sqrt{x-1} + 3$ up 2 units, followed by reflecting over the x-axis, followed by a horizontal shrink by a factor of $\frac{1}{3}$?

(A) $h(x) = -2\sqrt{3x-1} - 5$

(B) $h(x) = -2\sqrt{\frac{1}{3}x - 1} + 5$

(C) $h(x) = -2\sqrt{3x-3} - 5$

(D) $h(x) = 2\sqrt{-\frac{1}{3}x - 3} + 5$

7. If $f(x) = 5x + 1$ and $g(x) = \frac{x}{5} + k$ are inverse functions, what is the value of k?

(A) -1

(D) $-\frac{1}{5}$

(D) $\frac{1}{5}$

(B) 5

8. If $f(x) = \frac{5x-1}{x+2}$, find the equation of its inverse, $f^{-1}(x)$.

(A) $f^{-1}(x) = \frac{x+2}{5x-1}$

(B) $f^{-1}(x) = \frac{2x+1}{x-5}$

(C) $f^{-1}(x) = \frac{2x+1}{5-x}$

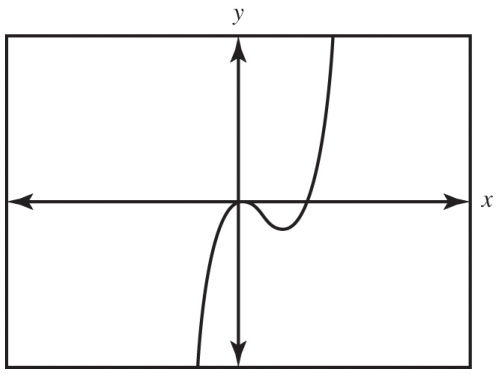
(D) $f^{-1}(x) = \frac{2x-1}{x-5}$

9. Cailey wants to buy a new outdoor heater. The cost for the unit is $349.99. If she plans to run the heater 5 months out of the year for an annual operating cost of $121.67, which function models the cost per

year over the lifetime of the unit, $C(t)$, in terms of the number of years, t, that she owns the heater?

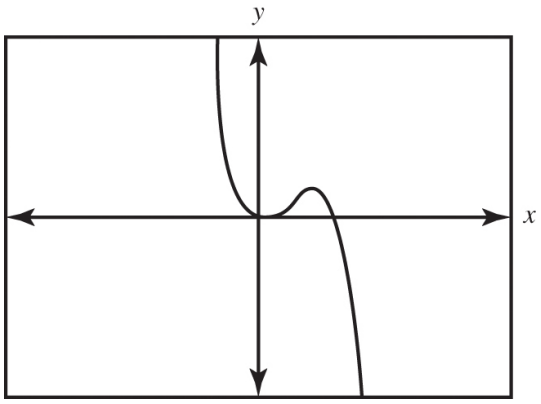
1. (A) $C(t) = 349.99 + 121.67t$
2. (B) $C(t) = 349.99 + 608.35t$
3. (C) $C(t) = \frac{349.99 + 121.67t}{t}$
4. (D) $C(t) = \frac{349.99 + 608.35t}{t}$

10. The accompanying graph represents the equation of the polynomial function $y = f(x)$.

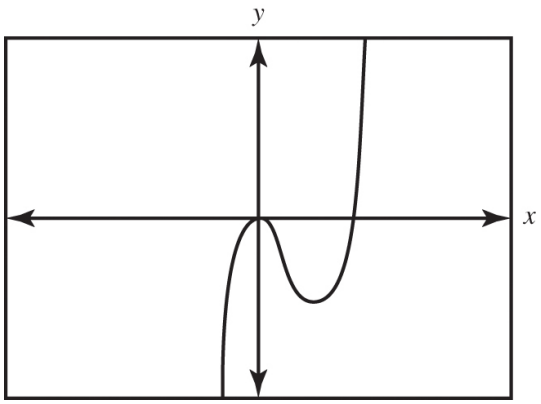


Which graph represents $g(x)$ if $g(x) = -f(x) + 3$?

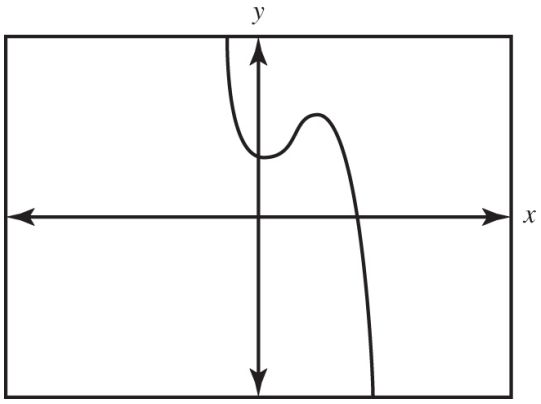
(A)



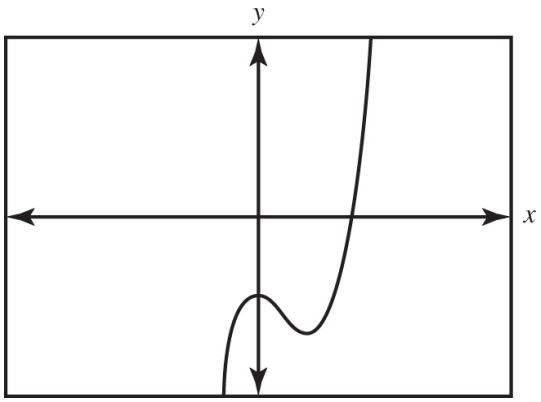
(B)



(C)



(D)



Answer Explanations

1. **(D)** To evaluate $f(a - 1)$, let the input, a - 1, be substituted into the function for x:

$$f(a - 1) = (a - 1)^2 + 1$$

Expand and combine like terms:

$$f(a-1) = (a-1)^2 + 1 = (a-1)(a-1) + 1$$
$$= a^2 - 2a + 1 + 1$$
$$f(a-1) = a^2 - 2a + 2$$

2. **(C)** The graph of a one-to-one function must pass both the vertical line test and the horizontal line test. Choice (A) fails the vertical line test at $x = 2$. Choice (B) fails the horizontal line test for any y-values between 0 and 1.25. Choice (D) fails the vertical line test for any $x > -2$. Choice (C) is the only graph that passes both the vertical and horizontal line test, and it can be recognized as the graph of the parent function $f(x) = \sqrt[3]{x}$.

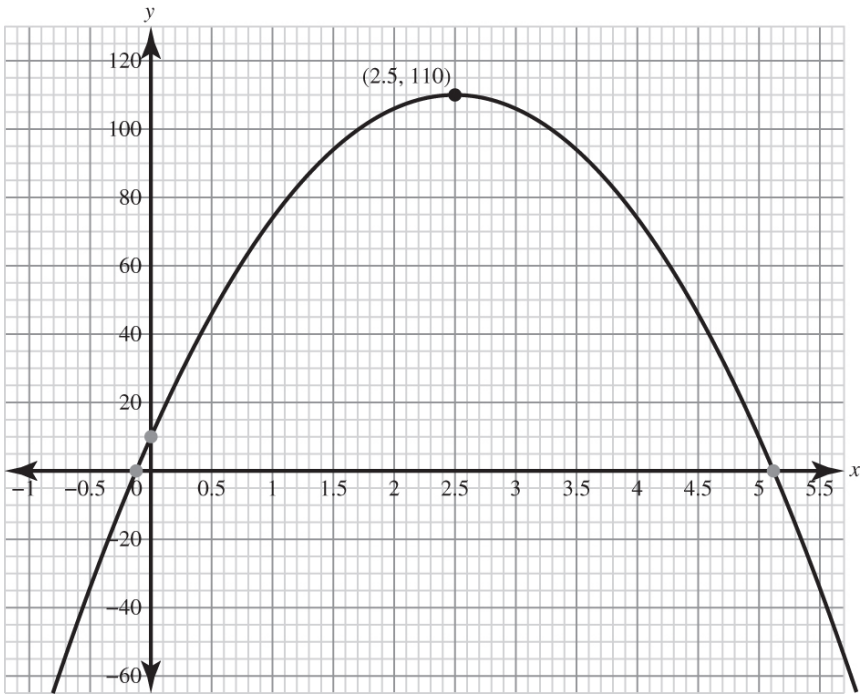
3. **(A)** If $f(x) = x^2 + 1$, then:

$$f(x+h) = (x+h)^2 + 1 = (x+h)(x+h) + 1$$
$$= x^2 + 2xh + h^2 + 1$$

Then:

$$\frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2xh + h^2 + 1 - (x^2 + 1)}{h}$$
$$= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h}$$
$$= \frac{2xh + h^2}{h} = 2x + h$$

4. **(B)** Since the function is a quadratic with a negative leading coefficient, the parabola is concave down, meaning there is a maximum point. Using a graphing calculator, the function can be graphed and the coordinates of the maximum point can be found. The y-coordinate of the point represents the maximum height.



The maximum height is 110 feet, and this occurs when the time is 2.5 seconds.

5. (C) First find the two composite functions separately.

$$f(g(x)) = f\left(\frac{x-1}{3}\right) = 3\left(\frac{x-1}{3}\right) + k = x - 1 + k$$

$$g(f(x)) = g(3x + k) = \frac{3x + k - 1}{3}$$

Then, set the two expressions equal and solve for k:

$$x - 1 + k = \frac{3x + k - 1}{3}$$

$$3(x - 1 + k) = 3x + k - 1$$

$$3x - 3 + 3k = 3x + k - 1$$

$$2k = 2$$

$$k = 1$$

6. (A) Each transformation affects the graph in a different way. A translation up 2 units is an addition of 2 to the function, a reflection over the x-axis negates the entire function, and a horizontal shrink by a factor of $\frac{1}{3}$ is a substitution of $3x$ in for x in the original function.

7. (B) If two functions are inverses of each other, their composition in either direction must equal x. For this problem, $f(g(x)) = g(f(x)) = x$. Find each composition, set them each equal to x, and solve for k:

$$f(g(x)) = f\left(\frac{x}{5} + k\right) = 5\left(\frac{x}{5} + k\right) + 1 = x + 5k + 1$$

$$x + 5k + 1 = x$$

$$5k + 1 = 0$$

$$k = -\frac{1}{5}$$

$$g(f(x)) = g(5x + 1) = \frac{5x + 1}{5} + k = x + \frac{1}{5} + k$$

$$x + \frac{1}{5} + k = x$$

$$\frac{1}{5} + k = 0$$

$$k = -\frac{1}{5}$$

8. (C) To find the equation of an inverse, reverse the x and the y and solve for y. The result will represent the inverse:

$$x = \frac{5y - 1}{y + 2}$$

$$x(y + 2) = 5y - 1$$

$$xy + 2x = 5y - 1$$

$$xy - 5y = -1 - 2x$$

$$y(x - 5) = -1 - 2x$$

$$y = f^{-1}(x) = \frac{-1 - 2x}{x - 5} = -\frac{1 + 2x}{x - 5} = \frac{1 + 2x}{-x + 5} = \frac{2x + 1}{5 - x}$$

9. (C) To determine the cost function, there is a fixed cost of $349.99 that happens once and then an additional operating cost of $121.67 that is paid every year. The operating cost is not dependent on the number of months Cailey runs the heater. To find the cost per year over the lifetime of the unit, take $349.99 + 121.67t$ and divide that by the number of years she owns the heater.

10. (C) The given function is $g(x) = -f(x) + 3$. Verbally, these changes to the function mean the graph is reflected over the x-axis and is shifted up 3 units. Only one graph has both transformations.

Unit 2

Exponential and Logarithmic Functions

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5

Exponential Functions

Learning Objectives

In this chapter, you will learn:

- ➜ The different properties of exponential functions
- ➜ How to graph exponential functions
- ➜ How to transform graphs of exponential functions
- ➜ How to model real-world scenarios using exponential functions

General Form

The general form of an exponential function is $f(x) = ab^x$, with the initial value a, where $a \neq 0$, and the base b, where $b > 0$ and $b \neq 1$.

The previous chapters dealt with algebraic functions, a function expressed as an equation in which real numbers and the variable are connected only by the operations of addition, subtraction, multiplication, division, and exponents. This chapter will discuss an important type of nonalgebraic function that can be used to model certain types of growth processes.

5.1 Exponential Functions

The following equations all have a 2 and an x: $y = 2x$, $y = x^2$, and $y = 2^x$. However, they are each very different functions. The first equation is a linear function, and its graph is a straight line. The second equation is a quadratic function, and its graph is a parabola. The third equation is called an exponential function. The exponential function has an exponent that is a variable, and its base is a number. An exponential function is a nonalgebraic function in which the independent variable is in the exponent.

Exponential Graphs

The functions $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ are both exponential functions. The graph of $f(x)$ is shown in Figure 5.1, and the graph of $g(x)$ is shown in Figure 5.2.

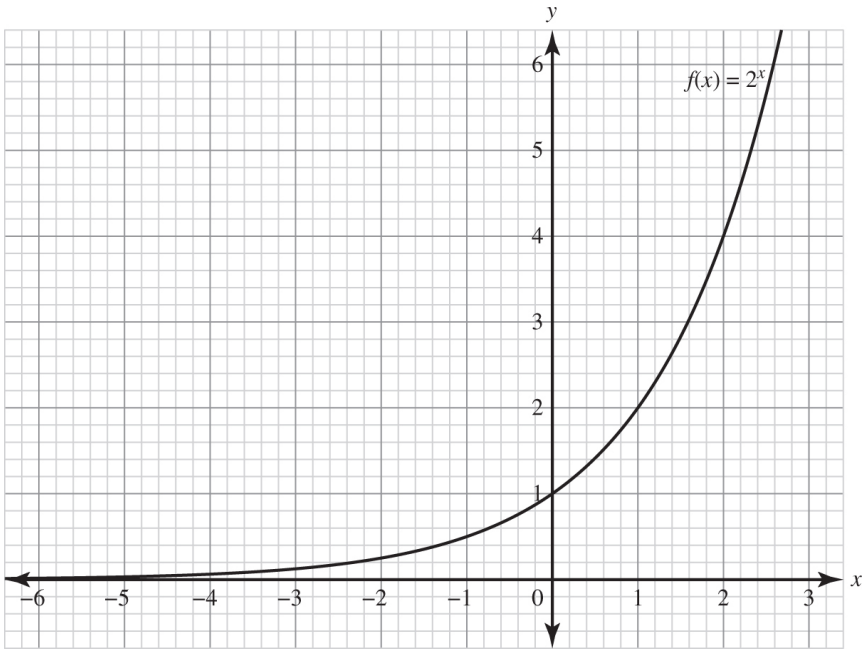


Figure 5.1

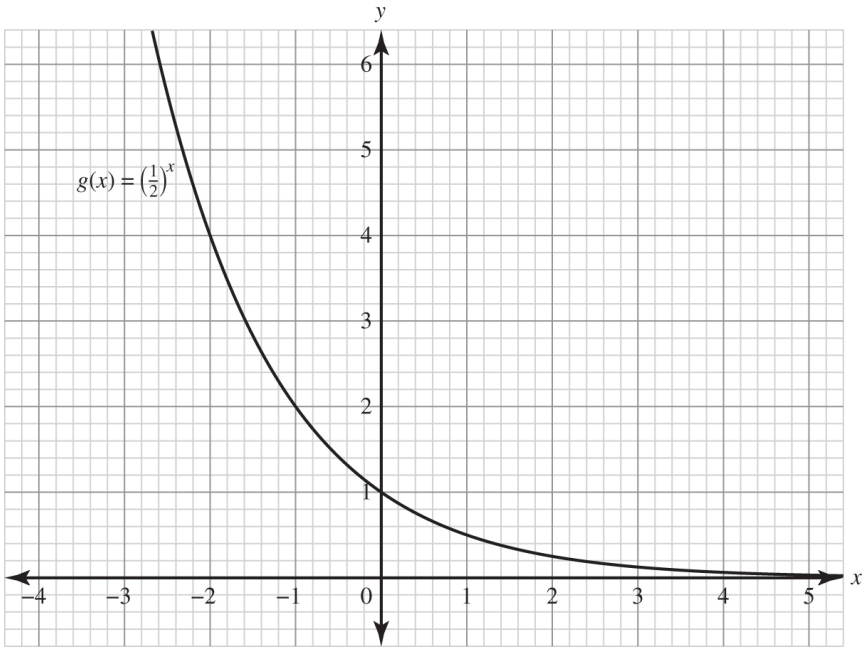


Figure 5.2

There are many similarities between the two graphs, which lead to generalizations for the graphs of exponential functions, $y = ab^x$:

1. Domain is all real numbers.
2. Range is $(0, \infty)$.
3. y-intercept is $(0, 1)$.
4. Horizontal asymptote at $y = 0$.
5. When $a > 0$ and $b > 1$, the function always increases, or demonstrates exponential growth.
6. When $a > 0$ and $0 < b < 1$, the function always decreases, or demonstrates exponential decay.
7. Exponential functions do not have extrema unless on a closed interval.
8. When $a > 0$ and $b > 0$, the graph is always concave up.
9. When $a < 0$ and $b > 0$, the graph is always concave down.
10. The graphs of exponential functions do not have inflection points.

Example

Use the graphs of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$, and the graph of $h(x) = -3^x$ shown in Figure 5.3, to evaluate the following limits.

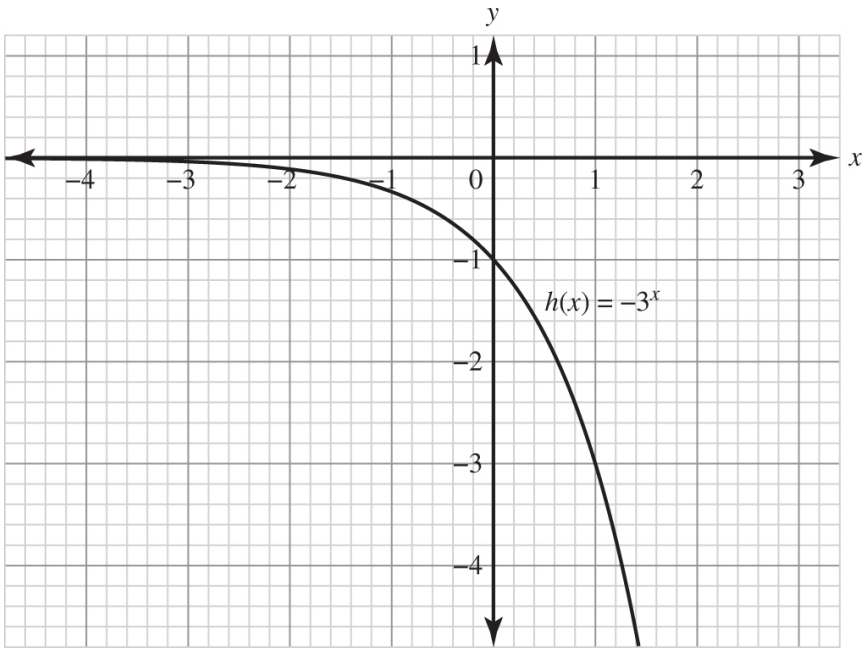


Figure 5.3

1. $\lim_{x \to -\infty} f(x)$

2. $\lim_{x \to \infty} f(x)$
3. $\lim_{x \to -\infty} g(x)$
4. $\lim_{x \to \infty} g(x)$
5. $\lim_{x \to -\infty} h(x)$
6. $\lim_{x \to \infty} h(x)$

✅ Solution

Since all these limits have x approaching infinity, to evaluate them, consider the end behavior of each function in either the negative or positive direction.

1. $\lim_{x \to -\infty} f(x) = 0$
2. $\lim_{x \to \infty} f(x) = \infty$
3. $\lim_{x \to -\infty} g(x) = \infty$
4. $\lim_{x \to \infty} g(x) = 0$
5. $\lim_{x \to -\infty} h(x) = 0$
6. $\lim_{x \to \infty} h(x) = -\infty$

End Behavior of Exponential Functions

For an exponential function in general form, as the input values increase or decrease without bound, the output values will increase or decrease without bound or will get arbitrarily close to 0.

This means that the end behavior of an exponential function can be one of three limits:

$$\lim_{x \to \pm\infty} a \cdot b^x = \infty$$

$$\lim_{x \to \pm\infty} a \cdot b^x = -\infty$$

$$\lim_{x \to \pm\infty} a \cdot b^x = 0$$

5.2 Exponential Function Manipulation

The familiar laws of integer and rational exponents hold true for variable exponents as well.

Laws of Exponents

- Multiplication (Product) Law of Exponents: $a^x \cdot a^y = a^{x+y}$
- Quotient Law of Exponents: $\frac{a^x}{a^y} = a^{x-y}$
- Power of a Power Law of Exponents: $(a^x)^y = a^{xy}$
- Negative Exponent Law: $a^{-x} = \frac{1}{a^x}$ and $\frac{1}{a^{-x}} = a^x$
- Rational Exponent Law: $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$
- Zero Exponent Law: $a^0 = 1$, $a \neq 0$

The laws of exponents lead to very interesting graphical transformations in exponential functions.

Example

Graph $f(x) = 2^x$ and $g(x) = 2^{x+3}$ on the same set of axes. State the transformation of the graph from $f(x)$ to $g(x)$.

Solution

The graphs of the two functions are shown in Figure 5.4 on the same set of axes.

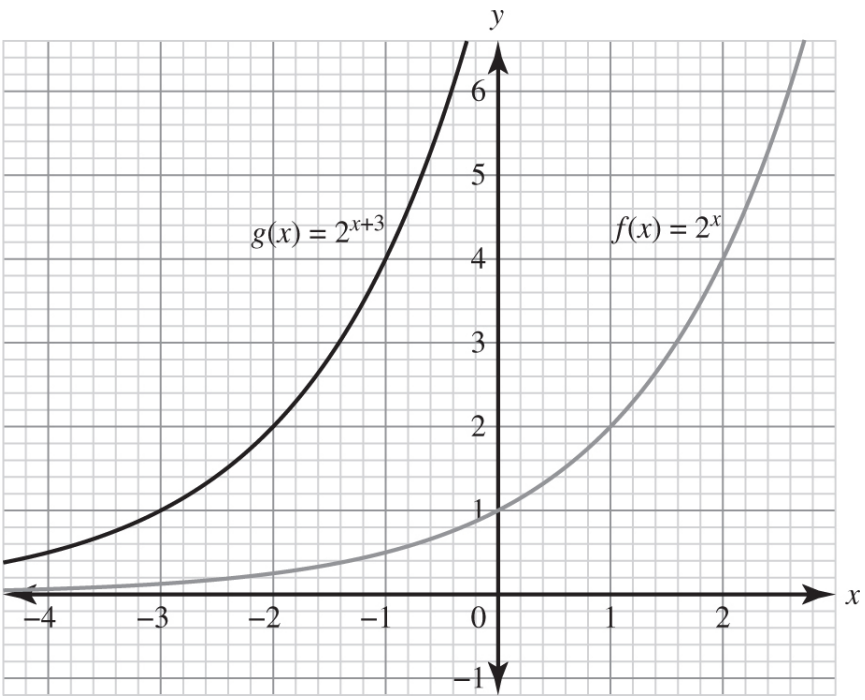


Figure 5.4

Since $g(x) = 2^{x+3} = f(x + 3)$, the $g(x)$ graph is the $f(x)$ graph shifted 3 units left.

This can also be looked at in a different way using laws of exponents. Since $g(x) = 2^{x+3} = 2^x \cdot 2^3 = 2^3 \cdot 2^x = 8 \cdot 2^x$, this can also be considered a vertical stretch of the graph of $f(x)$ by a factor of 8. Figure 5.5 is the graph of $f(x) = 2^x$ and $h(x) = 8 \cdot 2^x$.

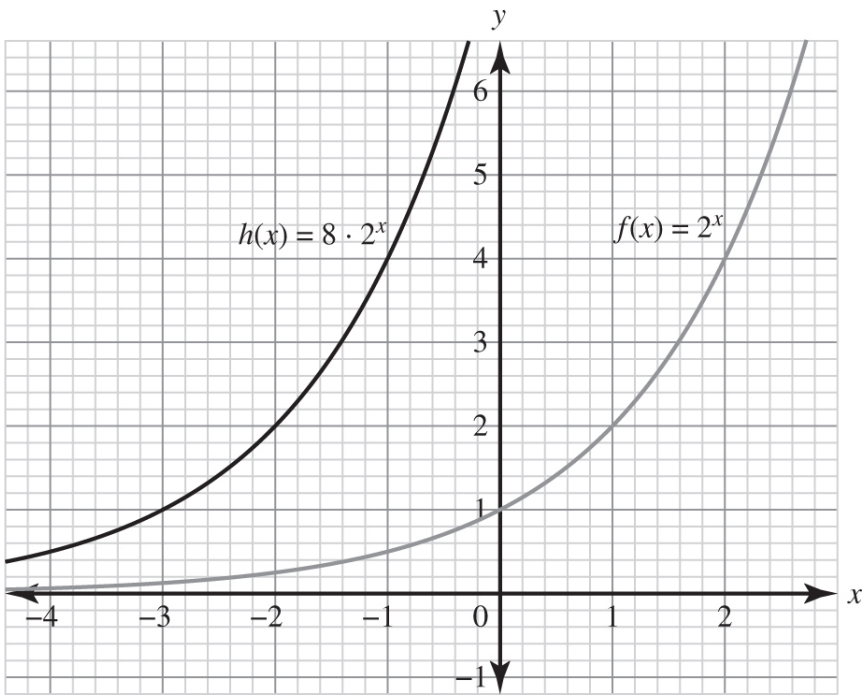


Figure 5.5

The graphs of $g(x)$ and $h(x)$ are identical.

Product Property and Vertical Dilation

The product property for exponents states that $b^m b^n = b^{(m+n)}$. Graphically, this property implies that every horizontal translation of an exponential function, $f(x) = b^{(x+k)}$, is equivalent to a vertical dilation, $f(x) = b^{(x+k)} = b^x b^k = ab^x$, where $a = b^k$.

Example

Given the exponential function $g(x) = 5^{x-1}$:

(a) State its parent function, $f(x)$.

(b) State the transformation that exists between the given function, $g(x)$, and its parent function, $f(x)$.

(b) Write a unique function, $h(x)$, that is equivalent to the function $g(x)$.

(b) State the transformation that exists between $h(x)$ and its parent function, $f(x)$.

✓ Solution

(a) The parent function is $f(x) = 5^x$.

(b) The graph of $g(x)$ is a horizontal shift 1 unit to the right of the graph of $f(x)$.

(b) Since $g(x) = 5^{x-1} = 5^x \cdot 5^{-1} = 5^{-1} \cdot 5^x = \frac{1}{5} \cdot 5^x$, $h(x) = \frac{1}{5} \cdot 5^x$.

(b) The graph of $h(x)$ is a vertical shrink of the graph of $f(x)$ by a factor of $\frac{1}{5}$.

The power of a power law of exponents also leads to an interesting graphical transformation.

Power Property and Change of Base

The power property for exponents states that $(b^m)^n = b^{(mn)}$. Graphically, this property implies that every horizontal dilation of an exponential function, $f(x) = b^{(cx)}$, is equivalent to a change of the base of an exponential function, $f(x) = (b^c)^x$, where b^c is a constant and $c \neq 0$.

› Example

Given the exponential function $g(x) = 81^x$, state the factor of the horizontal dilation of $g(x)$ if its parent function is $f(x) = 3^x$.

✓ Solution

Since $81 = 3^4$, then $g(x) = 81^x = (3^4)^x = 3^4x$. Therefore, the graph of $g(x)$ is a horizontal shrink by a factor of $\frac{1}{4}$ of the graph of $f(x)$.

› Example

Show that the graph of $g(x) = \frac{1}{4^x}$ is a reflection over the y-axis of the graph of $f(x)=4^x$.

✓ Solution

Using the negative exponent law, $g(x) = \frac{1}{4^x} = 4^{-x} = f(-x)$. $f(-x)$ reflects the graph of the original function, $f(x)$, over the y-axis.

Example

Rewrite the function $h(x) = \sqrt[5]{\frac{1}{3^x}}$ in the form $f(x) = ab^x$.

Solution

Using the exponent laws, $h(x) = \sqrt[5]{\frac{1}{3^x}} = \sqrt[5]{3^{-x}} = (3^{-x})^{\frac{1}{5}} = 3^{\frac{-x}{5}} = 1 \cdot 3^{\frac{-x}{5}}$.

5.3 Exponential Function Context and Data Modeling

Previous chapters discussed how to recognize if tables of data represented linear or quadratic functions. Exponential functions model growth patterns where successive output values over equal-length input value intervals are proportional. When the input values are whole numbers, exponential functions model situations of repeated multiplication of a constant to an initial value.

Example

Determine whether the functions in the tables below are linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

1.
2.
3.

Solution

For each table, the input values are of equal length, so calculate the average rate of change. If the average rate of change is constant, the function is linear. Also compute the ratio of consecutive outputs. If the ratio is constant, the function is exponential.

1. For Table 5.1, the average rate of change for every 1 unit increase in x is –3. Since this is a constant rate of change, the function is linear. To write the equation of the linear function, $y = mx + b$, find the slope, m, and the y-intercept, b. The average rate of change is the slope, so $m = -3$. The y-intercept, b, is the value of the function at $x = 0$, so $b = 2$. The linear function that models this data is $f(x) = -3x + 2$.

Table 5.1

x	$f(x)$
-1	5

x	$f(x)$
0	2
1	-1
2	-4
3	-7

2. For Table 5.2, the average rate of change from –1 to 0 is 2, and the average rate of change from 0 to 1 is 3. Because the average rate of change is not constant, the function is not linear. The ratio of consecutive outputs from –1 to 0 is 2, and the ratio of consecutive outputs from 0 to 1 is $\frac{7}{4}$. Because the ratio of consecutive outputs is not constant, the function is not an exponential function.

Table 5.2

x	$g(x)$
-1	2
0	4
1	7
2	11
3	15

3. For Table 5.3, the average rate of change from –1 to 0 is –16, and the average rate of change from 0 to 1 is –8. Because the average rate of change is not constant, the function is not linear. The ratio of consecutive outputs from –1 to 0 is $\frac{1}{2}$, and the ratio of consecutive outputs from 0 to 1 is $\frac{1}{2}$. The ratio of all consecutive outputs for every 1-unit increase is $\frac{1}{2}$. Since the ratio of consecutive outputs is constant, the function is an exponential function with a growth factor of $b = \frac{1}{2}$. The initial value of the exponential function, a, can be found when $x = 0$, so $a = 16$. Therefore, the exponential function that models the data is $h(x) = a \cdot b^x = 16 \cdot \left(\frac{1}{2}\right)^x$.

Table 5.3

x	$h(x)$
-1	32

x	$h(x)$
0	16
1	8
2	4
3	2

An exponential function model can be constructed from an appropriate ratio and initial value as shown in the previous example. An exponential function model can also be constructed from two input-output pairs. The initial value and the base can be found by solving a system of equations resulting from two input-output pairs.

Example

Determine an equation for the exponential function whose graph is shown in Figure 5.6.

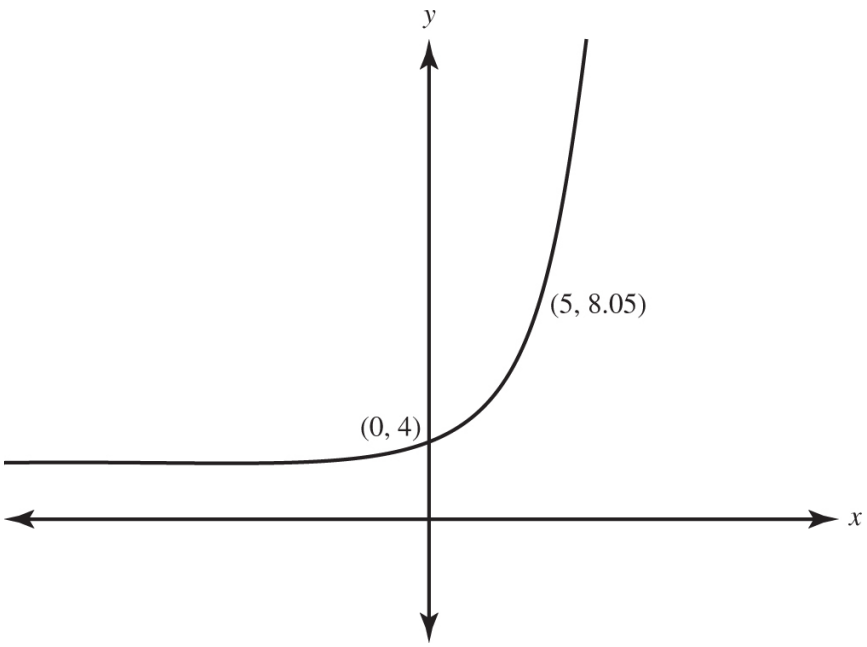


Figure 5.6

Solution

Since it was given that this models an exponential function, use the two sets of ordered pairs of the function to create a system of equations to find the a- and b-values in the general exponential equation form, $f(x) = a \cdot b^x$.
Substituting (0, 4) yields the equation $4 = a \cdot b^0$. Since $b \neq 0$, then $b^0 = 1$ and $a = 4$.

Substitute (5, 8.05) and $a = 4$ into the equation $f(x) = a \cdot b^x$:

$$8.05 = 4 \cdot b^5$$

Solve for b:

$$b^5 = \frac{8.05}{4}$$

$$b = \sqrt[5]{\frac{8.05}{4}} \approx 1.15$$

An equation to model the exponential function is $f(x) = 4 \cdot (1.15)^x$.

Modeling Growth and Decay

The growth patterns of many types of biological and physical quantities closely approximate exponential curves whose equations have the general form:

$$f(x) = a \cdot b^x$$

where a represents the initial amount, b represents the constant growth factor, and x represents the time variable.

When a quantity grows linearly over time, it increases by the same amount over equal intervals of time. When a quantity grows exponentially, it increases by a fixed percent of its previous value over equal intervals of time.

If a quantity is increasing at a constant annual rate of r of its previous value, after t years, an initial amount, a, of that quantity has grown to amount $f(t)$.

› Example

Suppose that $450 is deposited in a bank account that earns 8.2% interest compounded annually. State the balance in the account after each of the following time periods.

(a) 1 year

(b) 2 years

(b) t years

✓ Solution

(a) After 1 year, the balance in the account is

$$450 + (450)(0.082) = 450[1 + 0.082] = 450(1.082)$$

(b) After 2 years, the balance in the account is

$$450(1.082) + (450(1.082)) \cdot (0.082) = 450(1.082)[1 + 0.082] = 450(1.082)(1.082) = 450(1.082)^2$$

(c) Based on (a) and (b), a pattern is emerging that leads to a general formula for t years:

$$f(t) = 450(1.082)^t$$

The function in part (c), $f(t) = 450(1.082)^t$, is called an exponential growth model. The initial amount $a = f(0) = 450$, and the annual growth factor is $1 + r = 1 + 0.082 = 1.082$. If a quantity is decreasing at a constant annual rate of r% of its previous value, then $r < 0$, and the exponential function that describes the process is referred to as an exponential decay model.

Example

Determine the exponential function that has an initial value of 15 and increases at a rate of 12% each year.

Solution

The given information provides the initial value, $a = 15$, and the rate of increase, $r = 12\% = 0.12$. Therefore, the exponential function is $f(t) = 15\ (1.12)^t$, where t represents time in years.

Example

Suppose a culture of 100 bacteria in a Petri dish doubles every hour. Write a model that fits this bacterial growth, where $P(t)$ represents the total bacteria after t hours. Use the model to predict when the number of bacteria will be 5,000.

Solution

To recognize the type of model that is being described with the given information, create a table of values until a pattern is recognized. This has been done in Table 5.4.

Table 5.4

Total bacteria after 1 hour	$100 \cdot 2 = 200$
Total bacteria after 2 hours	$200 \cdot 2 = 400$ or $100 \cdot 2 \cdot 2 = 400$
Total bacteria after 3 hours	$400 \cdot 2 = 800$ or $100 \cdot 2 \cdot 2 \cdot 2 = 800$

An equation that models this behavior is $P(t) = 100 \cdot 2^t$.

To predict when the number of bacteria will be 5,000, use technology to graph the exponential model and the line $y = 5{,}000$ and find their point of intersection. This is shown in Figure 5.7.

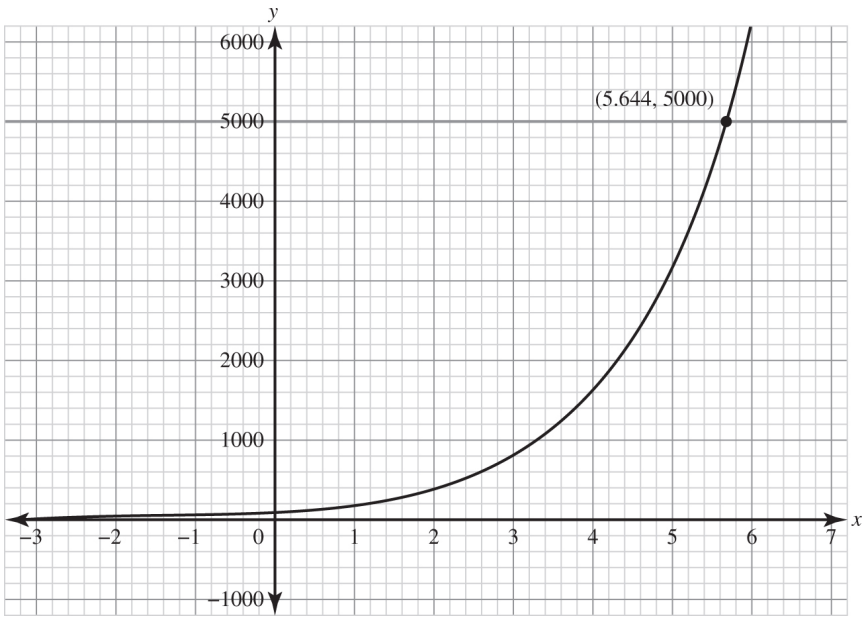


Figure 5.7

The point of intersection is (5.644, 5000). The number of bacteria will be 5,000 after 5.644 hours.

Example

Suppose the half-life of a certain radioactive substance is 20 days and there are 5 grams present initially. Write a function that models this situation, and use the model to predict the time when there will be 1 gram of the substance remaining.

Solution

If t is the time in days, the number of half-lives is $\frac{t}{20}$. The function $f(t) = 5\left(\frac{1}{2}\right)^{t/20}$ models the mass in grams of the radioactive substance at time t.

To find when there will be 1 gram of the radioactive substance left, use technology to graph the exponential model and the line $y = 1$ and find their point of intersection. This is shown in Figure 5.8.

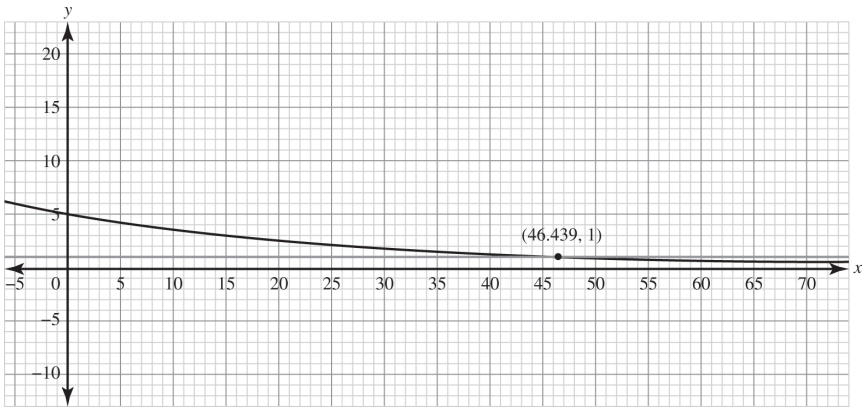


Figure 5.8

The point of intersection is (46.439, 1). There will be 1 gram of the radioactive substance left after approximately 46.439 days.

Example

Christopher buys a new car for $32,750. The car depreciates by about 11.75% per year. Write an equation to model this situation. Use the equation to determine the value of the car after 4 years.

Solution

The given situation can be modeled by an exponential decay function, $f(t) = a \cdot (1 + r)^t$, where $r = -11.75\% = -0.1175$, $a = 32{,}750$, t = years, and $f(t)$ = value of the car.

An equation to model the data can be written as $f(t) = 32{,}750 \cdot (1 - 0.1175)^t$.

The value of the car after 4 years can be found by evaluating $f(4) = 32{,}750\ (1 - 0.1175)^4 = \$19{,}864.16$.

Like with linear and quadratic functions, exponential function models can be constructed for a data set with technology using exponential regressions.

Example

The breaking strength in tons, y, of a steel cable with diameter x, in inches, is given in Table 5.5.

Table 5.5

x(inches)	0.5	0.75	1	1.25	1.5

y(tons)	9.85	14.8	31.3	48.2	61.4

Using technology, fit an exponential regression equation to the data, expressing the coefficients to the nearest thousandth. Using this regression model, estimate, to the nearest tenth of a ton, the breaking strength of a steel cable with a diameter of 1.75 inches.

✓ Solution

Enter the data into L1 and L2 by pressing STAT EDIT. After the data have been entered, press STAT, CALC, and 0:ExpReg.

The calculator gives the coefficients for a and b. The exponential function that models the data is approximated by $y = 3.848(6.933)^x$.

If $x = 1.75$ inches, then $y = 3.848(6.933)^{1.75} \approx 113.9848583$ tons. The breaking strength of a steel cable with a diameter of 1.75 inches, to the nearest tenth of a ton, is 114.0 tons.

Modeling Continuous Growth and Decay

Quantities that grow or decay continuously over time at an exponential rate can be modeled by the following function:

$$f(t) = a \cdot e^{kt}$$

where a is the initial amount, $f(t)$ is the amount present after t units of time, e is the natural base and equals approximately 2.718, and k is some constant specific to the particular growth or decay process that the function is modeling.

› Example

If $12,000 is invested for 7 years and interest is compounded continuously at an annual rate of 5%, what is the balance at the end of 7 years?

✓ Solution

Since the interest is compounded continuously, the function $f(t) = a \cdot e^{kt}$ can be used to model the data. It is also given that the initial amount, a, is 12,000, the annual rate, $r = k$, is 5% = 0.05, and the time, t, is 7. Substitute the values into the equation:

$$f(7) = (12{,}000) \cdot e^{(0.05)(7)} \approx \$17{,}028.81$$

5.4 Competing Function Model Validation

Depending on the patterns of a set of data of an event, it is possible to construct linear, quadratic, and exponential models that can help make predictions based on the event. Two

variables in a data set that demonstrate a slightly changing rate of change can be modeled by linear, exponential, and quadratic function models.

Models can be compared based on contextual clues, by considering scatterplots of data, or by examining the relationships with consecutive terms or rates of change to determine which model is most appropriate.

A model can be deemed appropriate by finding its coefficient of determination, which is notated as r^2. The r^2-value is a measure of the goodness of fit of a model. In regression, the r^2 coefficient of determination is a statistical measure of how well the regression predictions approximate the real data points. An r^2 of 1 indicates that the regression predictions perfectly fit the data. If using a TI calculator to find this value, press 2ND 0, and scroll down to DiagnosticOn. Press ENTER twice so that the word Done appears on the screen. Once this is complete, anytime a regression equation is found, the r^2-value will appear at the bottom of the regression model.

Example

Three different tables of data are given below. For each, determine whether a linear, quadratic, or exponential model is most appropriate for the data. Justify your answer. Using technology, write a regression equation that is a best fit model for each set of data.

1. Table 5.6 shows fuel efficiencies of a vehicle at different speeds.

Table 5.6

Miles per hour, x	20	24	30	36	40	45	50	56	60	70
Miles per gallon, y	14.5	17.5	21.2	23.7	25.2	25.8	25.8	25.1	24	19.5

2. Table 5.7 shows the humerus lengths (in centimeters) and heights (in centimeters) of several females.

Table 5.7

Humerus length (cm), x	22	25	26	27	28	30	32	33
Height (cm), y	130	142	141	145	152	154	159	166

3. Table 5.8 shows the U.S. population in millions between the years 1900–2011.

Table 5.8

Year, x	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000	2011
Population (in millions), y	76.2	92.2	106	123.2	132.2	151.3	179.3	203.3	226.5	248.7	281.4	311.6

☑ Solution

Since all the tables have x-values that are not equally spaced, the differences between the outputs cannot be analyzed. Instead, use technology to create a scatterplot of each set of data to find a function that models the data.

1. Figure 5.9 is the scatterplot for the data found in Table 5.6.

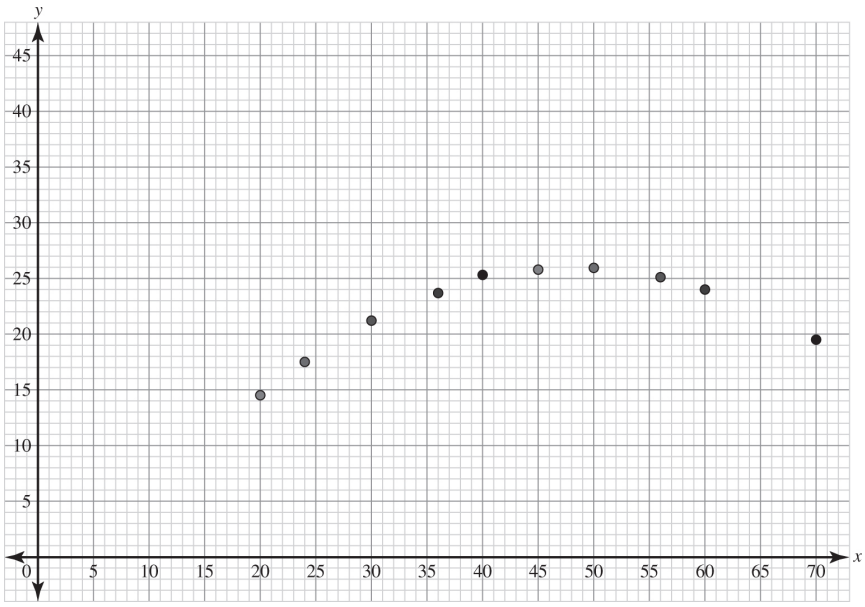


Figure 5.9

Based on the scatterplot, the data can best be modeled by a quadratic function.

Use a graphing calculator to find the best-fit equation by calculating the quadratic regression. The quadratic regression equation is $f(x) = -0.014x^2 + 1.366x - 7.144$ with $r^2 = 0.999$. Since the r^2-value is very close to 1, this indicates an appropriate model for the data set.

2. Figure 5.10 is the scatterplot for the data found in Table 5.7.

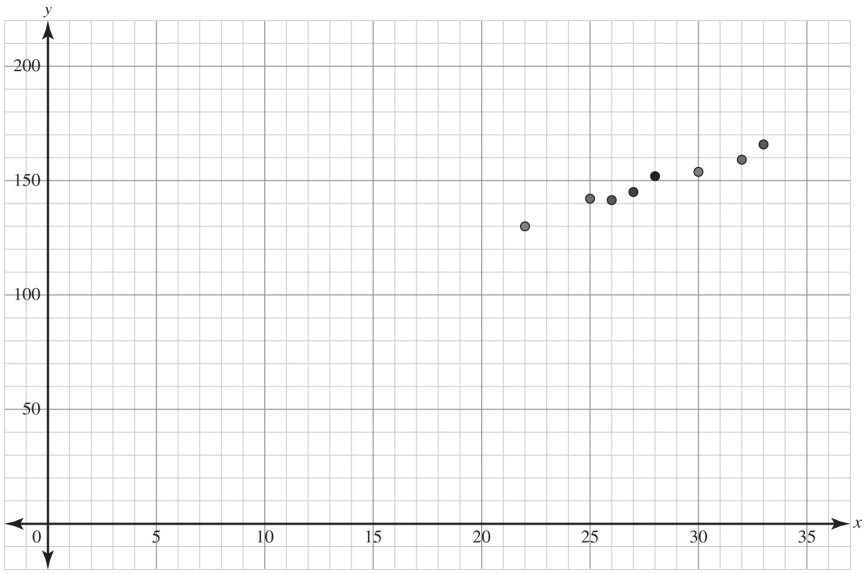


Figure 5.10

Based on the scatterplot, the data can best be modeled by a linear function.

Use a graphing calculator to find the best-fit equation by calculating the linear regression. The linear regression equation is $f(x) = 3.0527x + 63.531$ with $r^2 = 0.9847$. Since the r^2-value is very close to 1, this indicates an appropriate model for the data set.

3. When working with years, often it is easier to scale the years to have a better-fitting scatterplot. For the data in Table 5.8, let x represent the number of years after 1900. So in Figure 5.11, 0 represents 1900, 10 represents 1910, and so on.

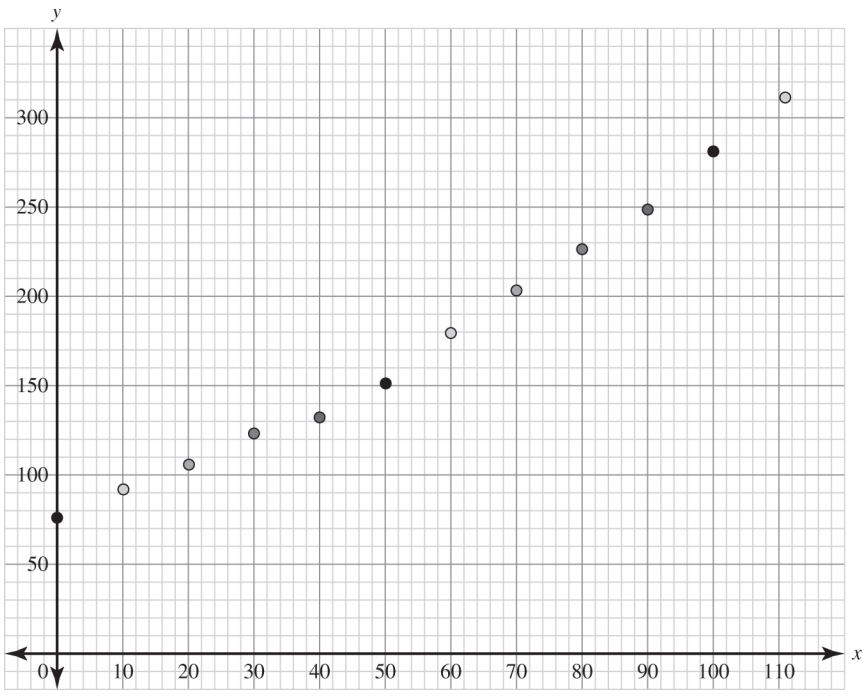


Figure 5.11

Based on the scatterplot, the data can best be modeled by an exponential function.

Use a graphing calculator to find the best-fit equation by calculating the exponential regression. Enter the data in the calculator using the scaled x-values. The exponential regression equation is $f(x) = 81.27455 \cdot 1.0126^x$ with $r^2 = 0.9946$. Since the r^2-value is very close to 1, this indicates an appropriate model for the data set.

Example

Table 5.9 shows the U.S. population in millions between the years 1900–2000. Using technology, find an exponential regression model that represents the values shown in the table. Use that regression equation to predict the U.S. population for the year 2011. Compare the result with the actual 2011 population, which was 311.6 million.

Table 5.9

Year, x	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000
Population (in millions), y	76.2	92.2	106	123.2	132.2	151.3	179.3	203.3	226.5	248.7	281.4

Solution

Enter the data into a calculator. The exponential regression equation is $f(x) = 80.55136 \cdot 1.01289^x$ with $r^2 = 0.995$, which indicates that this is a close fit.

To find the predicted population in the year 2011, evaluate

$$f(111) = 80.55136 \cdot 1.01289^{(111)} \approx 333.8$$

Compare this predicted value to the actual value in Table 5.8; the predicted value is an overestimation of the population by about 22 million persons. The difference between the predicted and the actual values is known as the error in the model.

This error can also be shown visually by graphing the regression equation with the scatterplot as shown in Figure 5.12.

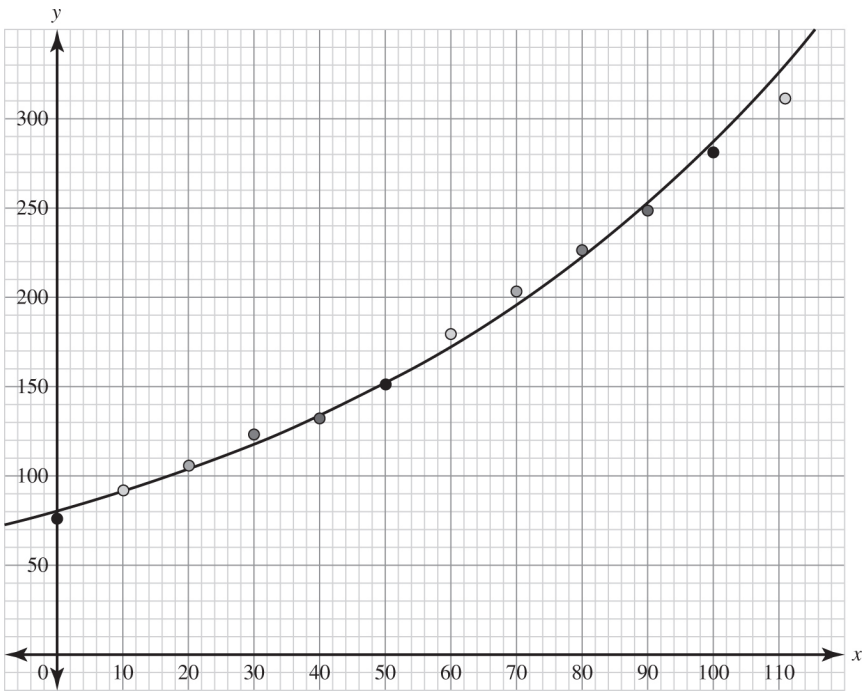


Figure 5.12

The graph of the regression equation is well above the data point (111, 311.6). The distance between the regression equation and the other data points is not as great as the distance between the equation and the last data point, showing that the greatest error occurs with the 2011 estimation.

Residuals

The difference between an observed value of data and the predicted value from a regression equation is called a residual. Each data point has one residual.

A residual plot is a graph that shows the residuals on the vertical axis and the independent variable on the horizontal axis. If the points in a residual plot are randomly

dispersed around the horizontal axis, a linear regression model is appropriate for the data; otherwise, a nonlinear model is more appropriate.

Calculating the Residual

Residual = Observed value - Predicted value

Example

Below are four residual plots. Using each graph, comment on the most appropriate model for each set of data.

1.

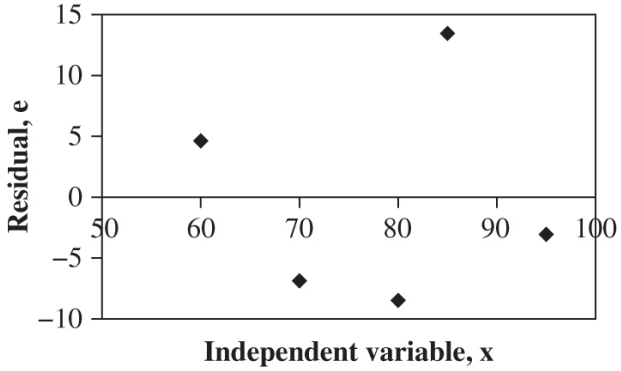


Figure 5.13

2.

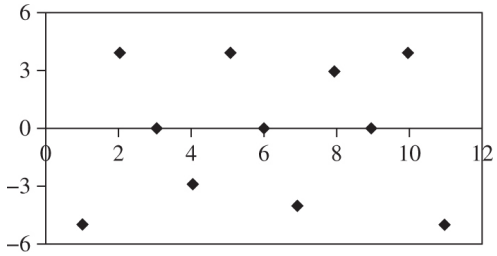


Figure 5.14

3.

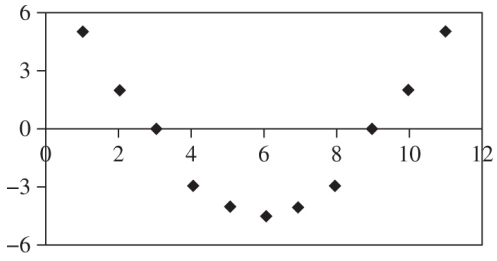


Figure 5.15

4.

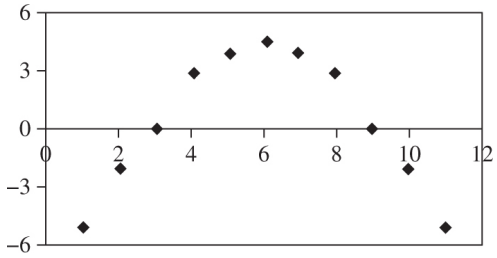


Figure 5.16

Solution

1. The plot in Figure 5.13 shows a random pattern. This random pattern indicates that a linear model provides an appropriate fit for the data.
2. The plot in Figure 5.14 shows a random pattern. This random pattern indicates that a linear model provides an appropriate fit for the data.
3. The plot in Figure 5.15 is U-shaped and not random. This nonrandom pattern indicates that a nonlinear model provides an appropriate fit for the data.
4. The plot in Figure 5.16 is U-shaped and not random. This nonrandom pattern indicates that a nonlinear model provides an appropriate fit for the data.

Example

Table 5.10 gives the years since 1890 and the population in millions of the people in California.

Table 5.10

Years since 1890, x	0	20	40	60	80	100
Population in millions, y	1.21	2.38	5.68	10.59	19.97	39.76

(a) Create a scatterplot using this data.

(b) Find the equation for the curve of best fit for the data.

(b) Create a residual plot, and comment on the appropriateness of your model from part (b).

Solution

(a) Figure 5.17 is a scatterplot using this data.

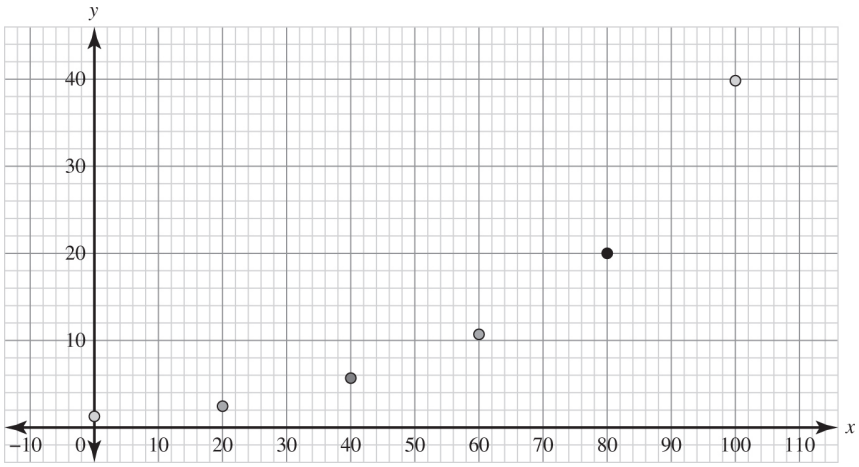


Figure 5.17

(b) Based on the scatterplot, it looks like an exponential regression equation would best fit the data. Using technology, the exponential regression equation is $y = 1.251579 \cdot (1.0355688)^x$ with $r^2 = 0.997558$.

(c) First create a table of values to help graph the residual plot, as shown in Table 5.11.

Table 5.11

x	y	Predicted y-value from Regression Equation	Residual = Observed value - Predicted value
0	1.21	1.2516	-0.0416
20	2.38	2.5179	-0.1379
40	5.68	5.0654	0.6146
60	10.59	10.191	0.399
80	19.97	20.501	-0.531

x	y	Predicted y-value from Regression Equation	Residual = Observed value - Predicted value
100	39.76	41.243	-1.483

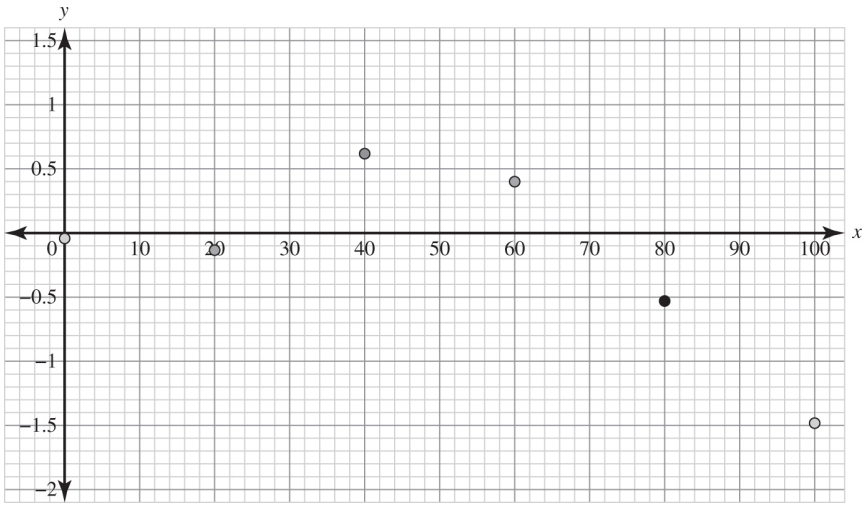


Figure 5.18

Overall, the residual plot shown in Figure 5.18 looks U-shaped. This nonrandom pattern indicates that a nonlinear model provides a more appropriate fit for the data. This would support using an exponential regression equation to model the data.

Multiple-Choice Questions

1. A scientist studying bacteria growth notices interesting behavior with the replication of cells over the course of 24 hours. She records her description of the behavior in hopes of writing a function that can be used to make predictions for future bacteria growth. She observed that for the first 8 hours, the number of bacteria, y, increased at a constant rate of 10 cells per minute. After the initial 8 hours, the number of bacteria varied inversely with time, x, with a proportional constant of 3 for the next 10 hours. Between hour 18 through hour 20, the bacteria decreased at a constant rate of 5 cells per minute. In the last 4 hours, the bacteria doubled in growth every minute. Which of the following models best fits the scientist's observations?

(A) $y = \begin{cases} 10x, 0 \le x \le 8 \\ \frac{3}{x}, 8 < x \le 18 \\ -5x, 18 < x \le 20 \\ 2^x, 20 < x \le 24 \end{cases}$

(B) $y = \begin{cases} 10x, 0 \le x \le 8 \\ \frac{3}{x}, 8 < x \le 18 \\ 5x, 18 < x \le 20 \\ 2x, 20 < x \le 24 \end{cases}$

(C) $y = \begin{cases} 10x, 0 \le x \le 8 \\ \frac{3}{x}, 8 < x \le 18 \\ 5x, 18 < x \le 20 \\ 2^x, 20 < x \le 24 \end{cases}$

(D) $y = \begin{cases} 10x, 0 \le x \le 8 \\ \frac{3}{x}, 8 < x \le 18 \\ -5x, 18 < x \le 20 \\ 2x, 20 < x \le 24 \end{cases}$

2. A radioactive substance has an initial mass of 250 grams, and its mass halves every 5 years. Which equation shows the number of grams, $M(t)$, remaining after t years?

(A) $M(t) = 250\ (5)^{t/4}$

(B) $M(t) = 250\left(\frac{1}{2}\right)^{t/5}$

(C) $M(t) = 250\ (5)^{-2t}$

(D) $M(t) = 250\left(\frac{1}{2}\right)^{5t}$

3. A culture of 3,000 bacteria triples every 40 minutes. If $P(t)$ represents the size of the bacteria population after t minutes, which equation can be used to model the exponential growth of these bacteria?

(A) $P(t) = 3{,}000\ (40)^{3t}$

(B) $P(t) = 3{,}000\ (3)^{t/40}$

(C) $P(t) = 3{,}000\ (3)^{40t}$

(D) $P(t) = 3{,}000\ (40)^{t/3}$

4. When exposed to sunlight, the number of bacteria in a culture decreases exponentially at the rate of 10% per hour. What is the best approximation for the number of hours required for the initial number of bacteria to decrease by 50%?

(A) 6.6

(B) 6.0

(C) 5.4

(D) 4.6

5. Which of the functions are equivalent?

I. $f(x) = (2)^{3-x}$

II. $g(x) = 8 \cdot 2^{-x}$

III. $h(x) = \frac{8}{2^x}$

(A) I and II only
(B) I and III only
(C) II and III only
(D) I, II, and III

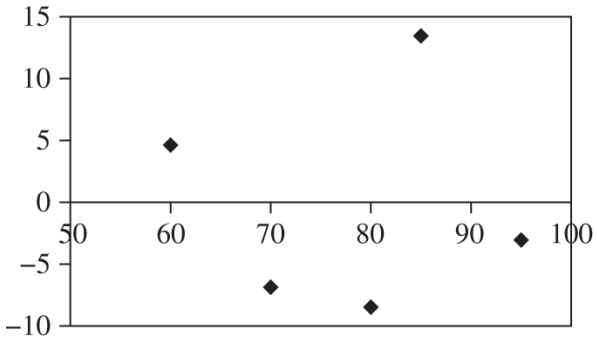
6. Which of the functions given below are equivalent?

I. $f(x) = (9)^{x/2}$
II. $g(x) = 3^x$
III. $h(x) = 4.5^x$

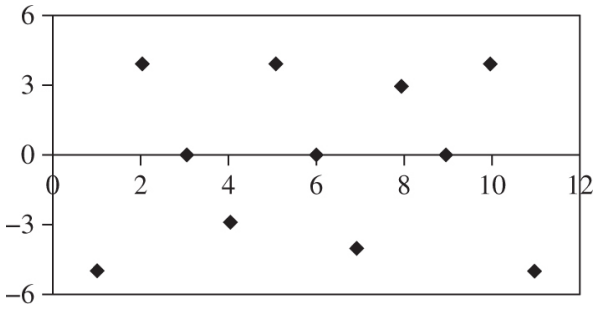
(A) I and II only
(B) I and III only
(C) II and III only
(D) I, II, and III

7. Michael buys a new computer at full price. He learns that each year the value of the computer depreciates at a rate of 7%. Which of the following residual plots would best fit the model of depreciation?

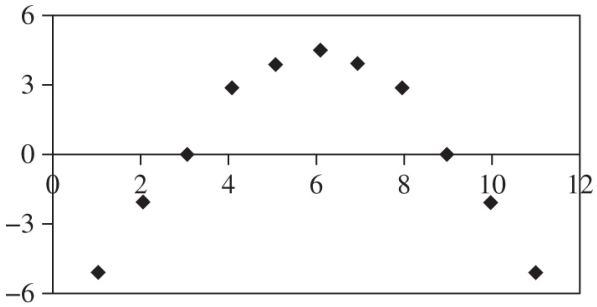
(A)



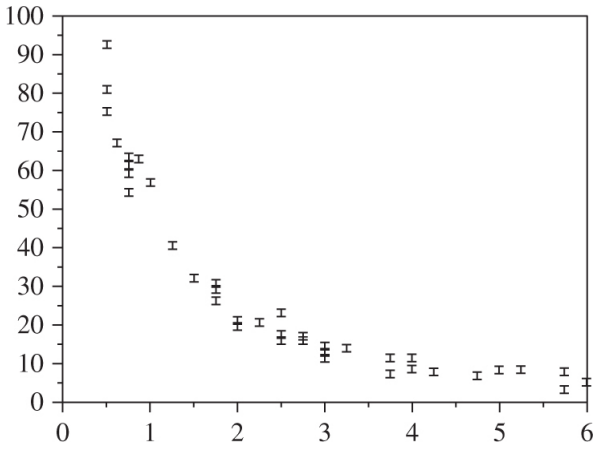
(B)



(C)



(D)



8. Which of the following tables of data best fits an exponential regression model?

(A)

x	-2	-1	0	1
y	.08	.4	2	10

(B)

x	-2	-1	0	1	2
y	2	4	4.8	5.5	6

(C)

x	-3	-2	-1	0	1
y	-7	-5	-3	-1	1

(D)

x	-2	-1	0	1	2
y	8	2	0	2	8

9. If $a > 0$, $b > 1$, and $c < 0$, evaluate $\lim_{x \to \infty} a \cdot b^{cx}$.

   (A) $-\infty$
   (B) 0
   (C) 1
   (D) ∞

10. Given $f(t) = -5 \cdot e^2 t$, find $\lim_{t \to \infty} f(t)$.

   (A) $-\infty$
   (B) 0
   (C) 1
   (D) ∞

Answer Explanations

1. **(A)** Look at the different time intervals. In the first 8 hours, the bacteria grew at a constant rate of change, which implies that it is a linear model with a slope of 10. The first equation should be $y = 10x$. In the second interval, between hours 8 and 18, the growth is described as an inverse variation, which takes on the model $y = \frac{k}{x}$. Since the constant of proportionality is given as 3, the equation is $y = \frac{3}{x}$. Between hours 18 and 20, the bacteria decreased at a constant rate of change, implying that a linear model with a negative slope best fits; $y = -5x$. Finally in the last 4 hours, the cells doubled, which means they grew in such a way that it was 2 then 4 then 8 then 16. This pattern follows the equation $y = 2^x$.

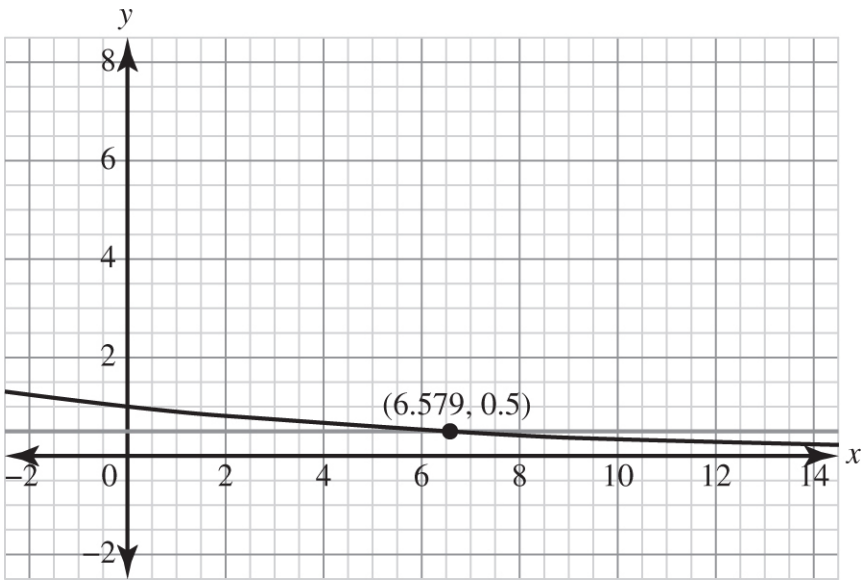
2. **(B)** If t is the time in years, the number of half-lives is $\frac{t}{5}$. The function $M(t) = 250\left(\frac{1}{2}\right)^{t/5}$ models the mass in grams of the radioactive substance at time t.

3. **(B)** If t is the time in minutes, the number of times the bacteria triples is $\frac{t}{40}$. The function $P(t) = 3{,}000\ (3)^{t/40}$ models the size of the bacteria population after t minutes.

4. **(A)** Since the model is exponentially decreasing, an equation to fit this data is $f(t) = a(1 - r)^t$. Let $r = 10\% = 0.10$, $a = 1$, and $f(t) = \frac{1}{2}$:

$$\frac{1}{2} = 1\,(1 - 0.10)^t$$

   Using technology, graph both $y = \frac{1}{2}$ and $y = 1\ (1 - 0.10)^t$ on the same set of axes and then find their point of intersection.



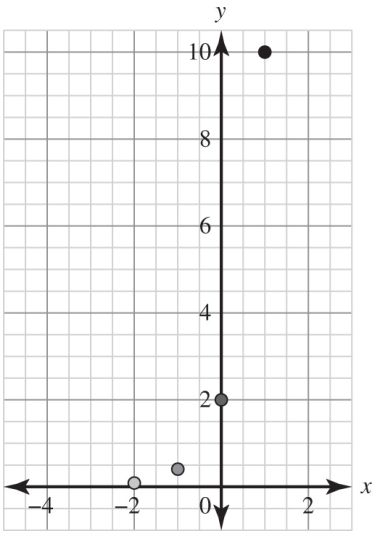
The point of intersection is (6.579, 0.5). The time the bacteria will be half of its initial value is approximately 6.579 hours.

5. (D) Beginning with the first equation, $f(x) = (2)^{3-x}$ and using the laws of exponents, $(2)^{3-x} = 2^3 \cdot 2^{-x} = 8 \cdot 2^{-x} = g(x)$. So I and II are equivalent. Using the law of negative exponents, 8. $2^{-x} = \frac{8}{2^x} = h(x)$. So I, II, and III are all equivalent.

6. (A) Since $3 = 9^{1/2}$, then $3^x = (9^{1/2})^x = 9^{x/2}$. So I and II are equivalent. There is no connection between base numbers 4.5 and 3 and 9, so III is not equivalent to the other two.

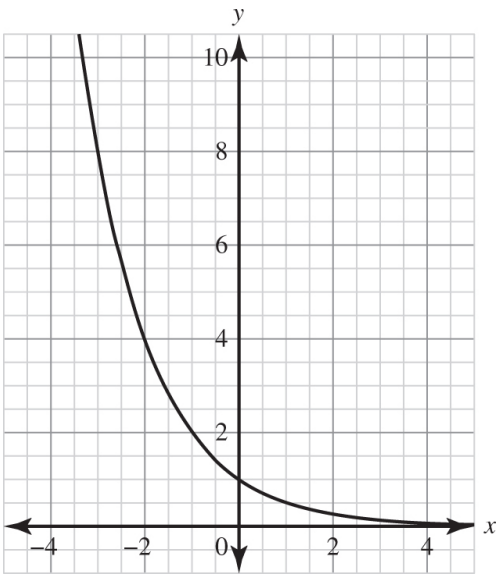
7. (C) The description of the situation fits an exponential model. Choices (A) and (B) show a random pattern. This random pattern indicates that a linear model provides an appropriate fit. The residual plot in Choice (C) is U-shaped and not random. This nonrandom pattern indicates that a nonlinear model provides a more appropriate fit for the data. The plot in Choice (D) is an exponential scatterplot, which is most likely what the graph of the data would look like. However, the question is asking about what the residual plot would look like.

8. (A) One way to choose a best-fit model is to use technology to create scatterplots to see the overall pattern of the data.



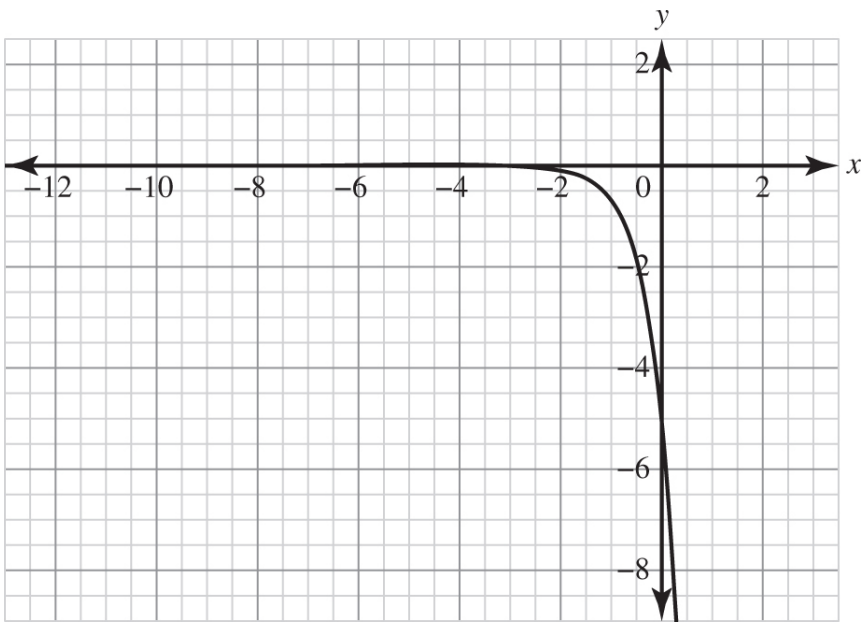
The scatterplot for Choice (A) follows an exponential model.

9. **(B)** To evaluate $\lim_{x \to \infty} a \cdot b^{cx}$, think of the graph of the function $y = a \cdot b^{cx}$. Since a and b are both positive and $b > 1$, and c is negative, it will follow the behavior of an exponential graph that is reflected over the y-axis, as shown below.



Therefore, $\lim_{x \to \infty} a \cdot b^{cx} = 0$.

10. **(A)** Graph the function $f(t) = -5 \cdot e^{2t}$.



Graphically, $\lim_{t \to \infty} f(t)$ approaches $-\infty$.

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$$a_6 = 1 + 2(6 - 1) = 1 + (2)(5) = 1 + 10 = 11$$

$$a_n = a_1 + d(n - 1)$$

$$a_n = a_0 + d(n)$$

$$a_n = a_k + d(n - k)$$

$$\begin{aligned} a_{11} &= a_{50} + (-3.5)(11 - 50) \\ 7 &= a_{50} + (-3.5)(-39) \\ 7 &= a_{50} + 136.5 \\ a_{50} &= -129.5 \end{aligned}$$

✓

$$-2, _____, _____, _____, _____, 13$$

$$\begin{aligned} a_6 &= a_1 + d(6-1) \\ 13 &= -2 + d(6-1) \\ 13 &= -2 + 5d \\ 15 &= 5d \\ d &= 3 \end{aligned}$$

$$\begin{aligned} a_2 &= -2 + 3 = 1 \\ a_3 &= 1 + 3 = 4 \\ a_4 &= 4 + 3 = 7 \\ a_5 &= 7 + 3 = 10 \end{aligned}$$

$$\frac{18}{6} = \frac{54}{18} = \frac{162}{54} = \ldots = 3$$

$$g_n = g_1 r^{n-1}$$
$$g_n = g_k r^{n-k}$$

$$g_n = g_0 r^n$$

›

$$\frac{64}{9}$$

✓

$$g_5 = \frac{64}{9}$$

$$\frac{64}{9} = 36r^{5-1}$$
$$\frac{64}{9} = 36r^4$$
$$r^4 = \frac{16}{81}$$
$$r = \pm\sqrt[4]{\frac{16}{81}} = \pm\frac{2}{3}$$

$r = -\frac{2}{3}$

$$g_6 = (36)\left(-\frac{2}{3}\right)^{6-1}$$
$$g_6 = -\frac{128}{27}$$

$r = \frac{2}{3}$

$$g_6 = (36)\left(\frac{2}{3}\right)^{6-1}$$
$$g_6 = \frac{128}{27}$$

$-\frac{128}{27}$ $\frac{128}{27}$

›

✓

$r = \frac{5}{1} = \frac{25}{5} = 5$

$$\frac{\Delta y}{\Delta x}$$

$$\frac{4-(-1)}{2-1}=\frac{5}{1}=5$$

$$\frac{9-4}{3-2}=\frac{5}{1}=5$$

$$\frac{5-1}{2-1}=\frac{4}{1}=4$$

$$\frac{25-5}{3-2}=\frac{20}{1}=20$$

$$g(n) = \left(\frac{1}{2}\right)^n$$

$$r = \frac{1}{2}$$

›

✓

›

$\left\{15, 5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}\right\}$

✓

$r = \frac{1}{3}$

$g_n = 15\left(\frac{1}{3}\right)^{n-1}$

$$\left\{15, 5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}\right\}$$

■	■
■	■

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L V [W F K R H R U G L Q D W H R I W K H S R L Q W
6 L Q F H L W L Q Y R O Y H V P R U H W K D Q R Q
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D Q G W K H Q V X E W U D F W D J D L Q W

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L V \ W F K R H R U G L Q D W H R I W K H S R L Q W I
6 L Q F H W K H D Q J O H L V U R W D W L 7 R J H L Q D
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L V \ W F K R H R U G L Q D W H R I W K H S R L $\theta = \frac{13\pi}{3}$
6 L C $\frac{13\pi}{3} > 2\pi$ L W L Q Y R O Y H V P R U H W K D Q R Q
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• I U R P W K 7 H R D L Q Q J G H W K H F R W H U $\frac{13\pi}{3} - 2\pi = \frac{7\pi}{3}$ D
D Q G W K H Q V X E V $\frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$ W D J D L Q W R J H W

$$\sin\left(\frac{13\pi}{3}\right) = \sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

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FDQ EH ILJXUHG RXW XVLQJ UHIOHFWLF
IURP 4XDGUDQW ,

6LQH DQG &RVLQH) XQFWLRQ* UDSKV

7UL JRQRPHWULF IXQFWLRQV DUH SHULRG DURXQG WKH XQLW FLUFOH RYHU UHJXODU LQWHUYDOV (DFK F\FOH YDULHV IURP WR DQG UHSHDWV RQ YDOXHV IRU SUHYLRXV VHFWLRQ VKRZHG WKDW DQ D PXOWLSOH RI VDLG LQG IRU VLQH DQG IXQFWLRQV ZLWK SHULRGV RI

7R GLVFRYHU WKH JUDSKV RI WKH VLQH FLUFOH WR SORW WKH YDOXHV RI VLQH

$V SR[QWP RYHV FRXQWHUFORFNZLVH L GHWHUPLQH.G VE\ WKH

)RU DQ\ JLYHQ SHFLILHV WR 3HKEV KRUL]RQWDO[7 SRVLWLRQV RI FUHD.WLQQDQ I... F.RLV EDVHG RQ..ZLWK WKH FLWDWU.RUS.F RW HYDOXDWLRQV DQG ZULWLQJ WKHP DV [S FRRUGLQDW.HDQHGS WKH HYDOXH

8VLQJ WKH XQLW FLUFOH DQG FKDUW EHHQ FUHD IXQFWLRQ V.L Q

7DEOH

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$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$
•	
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$
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7D E O H

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$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
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$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	

I	VLQ
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$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$
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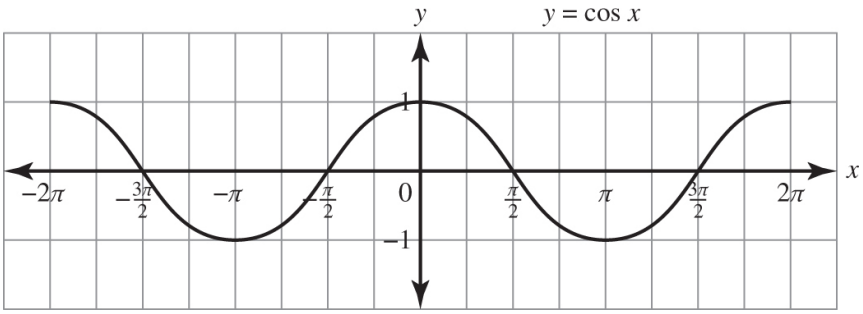
3OR WWLQJ WKH SRLQWV IURP HDFK WDE
JUDSK V RI RVR U. F R VDQ G VLQ RU. V LQDV V
LQ L J X U H V D Q G



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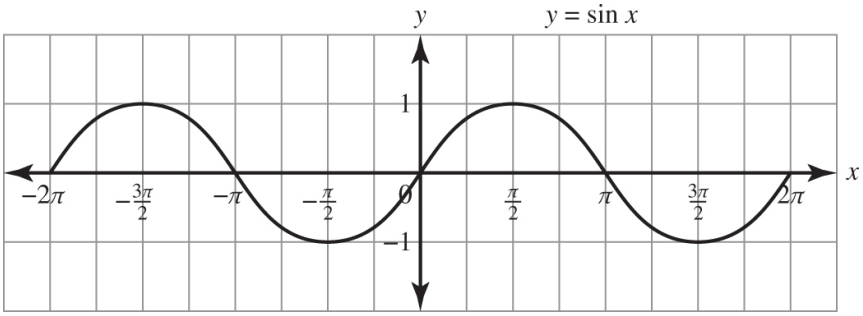

) L J X U H

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WUDQVIRUPDWLRQ RI WKH XQLW F XUYH W
7KH GRPDLQ RI ERWK WKH FRVLQH DQG
QXPEHUV UDQJH RI ERWK WKH FRVLQH D
7KH SHULRG RI ERWK WKH FRVLQH DQG V

6LQXVRLGDO)XQFWLRQV

$ VLQXVRLGDO IXQFWLRQ LV DQ\ IXQFWL
PXOWLSOLFDWLYHI.WUDVQVKRHUPDWHRDQVG
DUH H[DPSOHV RI D VL(cosθ=sin(θ+π2)DOZKXQKWQL
KRUL]RQWDO VKLIW OHIW E\

7KHUH DUH QR VLQH\ FKDUDFWHULVWLF
IXQFWLRQV

3HULRG DQG)UHTXHQF\

,Q JHQHUDO WKH SHULRG DQG IUHTXH
UHFLSURFDOV RI HDFK RWKHU VR LI QJR. I FR.VLV •
DQG WKH IUHTXH1/2π)F\ RI HDFK LV

)UHTXHQF\

7KH IUHTXHQF\ RI D VLQXVRLGDO IXQFWL
FRPSOHWHV LQ DQ LQWHUYDO RI • UDGL
FXUYH HDFK FRPSOHWH RQH F\FOH HYHU
FXUYH LV

$PSOLWXGH

7KH DPSOLWXGH RI D VLQXVRLGDO IXQFWLRQ LV KDOI WKH GLIIHUHQFH EHWZ
PD[LPXP DQG PLQLPXP YDOXHV 6LQFH W
RI ERWK WKH VLQH FXUYH DQG FRVLQH WKHLU
DPSOLWXGH. RI V.EDQg(θ)=cosθis1−(−1)2=1

0LGOLQH

7KH PLGOLQH RI WKH JUDSK RI D VLQXVRI
DYHUDJH RI WKH PD[LPXP DQG PLQLPXP
PD[LPXP DQG PLQLPXP YDOXHV RI ERWK

DQG U H V W U L F W L R Q V R Q W K H G R P D L Q R I
$g(\theta) = \cos\theta$ is $\frac{1+(-1)}{2} = 0$ 7 K H P L G O L Q H L V W K H K R U L] R

& R Q F D Y L W \

$ V L Q S X W Y D O X H V L Q F U H D V H W K H J U D S
E H W Z H H Q F R Q F D Y H G R Z Q D Q G F R Q F D Y H

(Y H Q 2 G G) X Q F W L R Q V

6 L Q F H W K H G R P D L Q R I S R L Q W V \ P P H W U \ Z L
L W L V D Q R G G H U I X Q F W L R Q 99 V L Q ...

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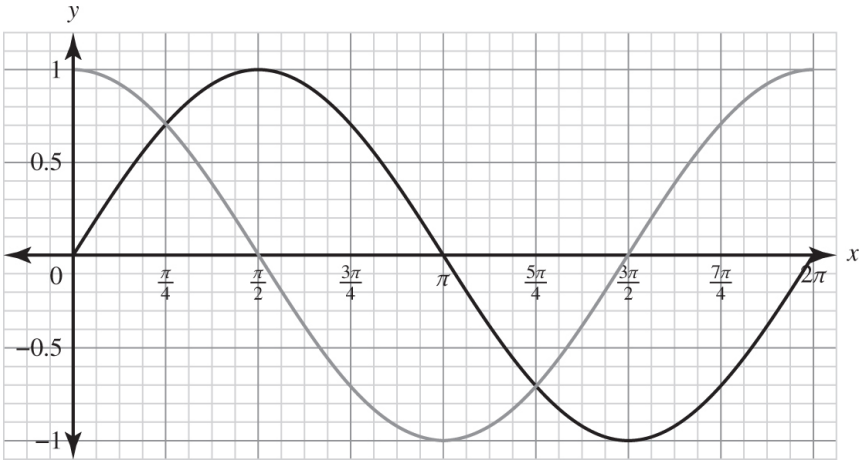
G H F U H D V L Q J D Q G D W W K H V D P H L Q W H U Y D O

6 R O X W L R Q

7R J H W D Y L V X D O V \ N H W F K Q G H U Y D O V R I D [H V D V) L J X U H



) L J X U H

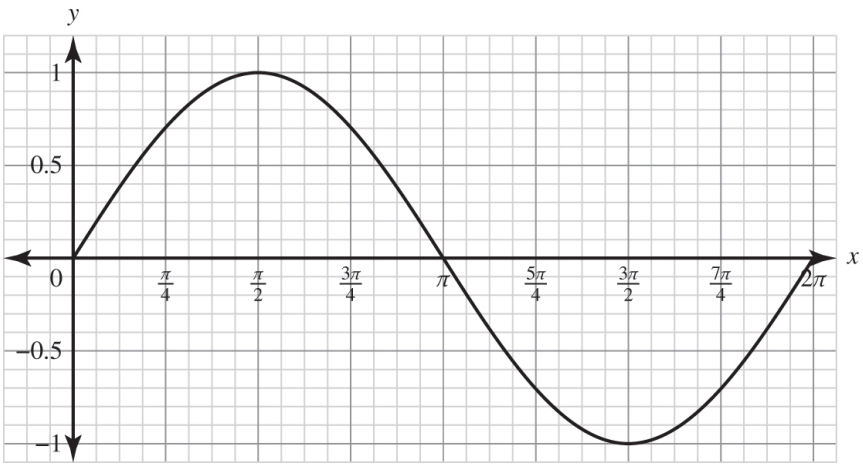
7KH VLQH FXUYH LV GHFUHDVLQJ DQG WI
LQW $\left(\pi, \frac{3\pi}{2}\right)$ Y DO

([DPSOH

8VH WKH JUDSKLQJ RIFRVLQ WKH LQWHUY\DO
VL[QDQG FRVLQ WKH LQWHUYDO >9 • •@

6RO XWLRQ

7KH JUDSKVRQ WKH LQWHUYDO LJXUH •@ L



)LJXUH

6LQFH VLQH LV DQ RGG IXQFWLRQ WKH
WUDQVIRUPHG VXFK WKDW ZKR ZO LQ G
ZKLFK JLYHV WKH UHVXOWV RI IROORZLQ

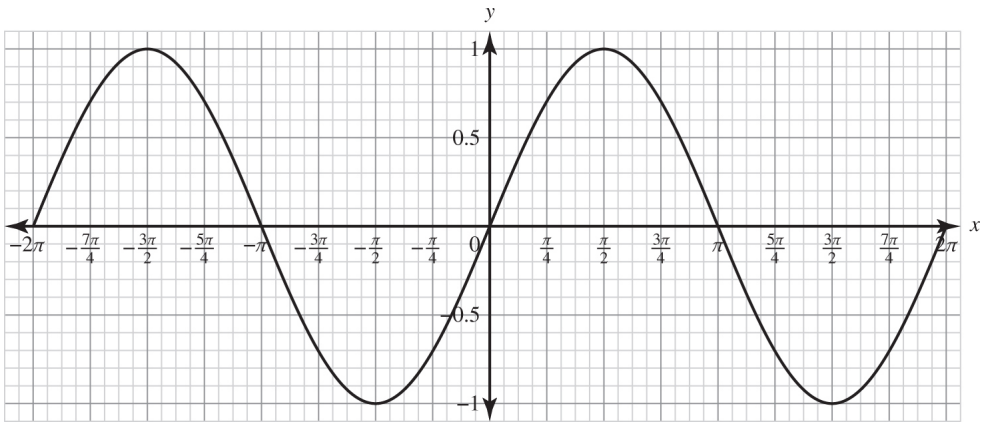
7DEOH

[		$\frac{\pi}{2}$	•	$\frac{3\pi}{2}$	•
\					

7D E O H

[		$-\frac{\pi}{2}$	9	$-\frac{3\pi}{2}$	9
\					

3 O R W W L Q J W K H S R L Q W V J X U H D G V W R W K H



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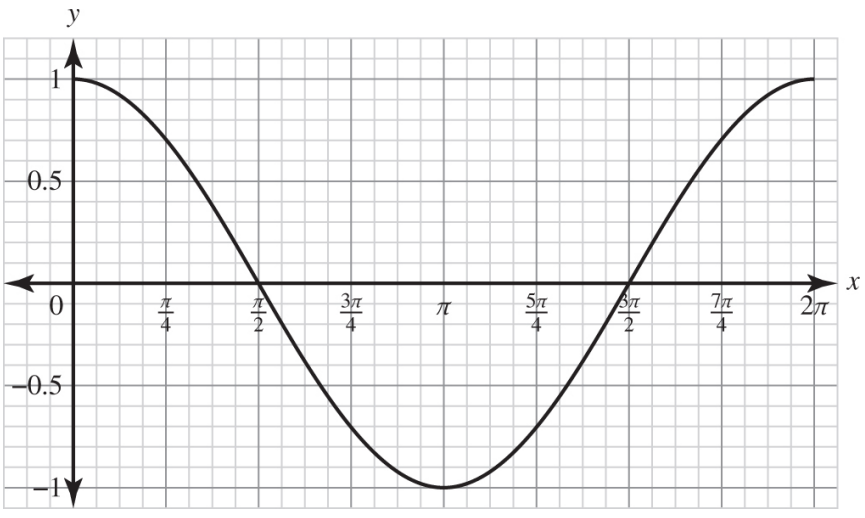

)LJXUH

6LQFH FRVLQH LV DQ HYHQ IXQFWLRQ WKH YDOXHV RI WKH WUDQVIRUPHG FRVLQH IXQFWLRQ DW [DQG [DUH WKH VDPH 7DEOH DQG ZKLFK LV DQ RGG ZKLFK JLYHV WKH UHVXOWV RI IROORZLQ

7DEOH

[		$\frac{\pi}{2}$	•	$\frac{3\pi}{2}$	•
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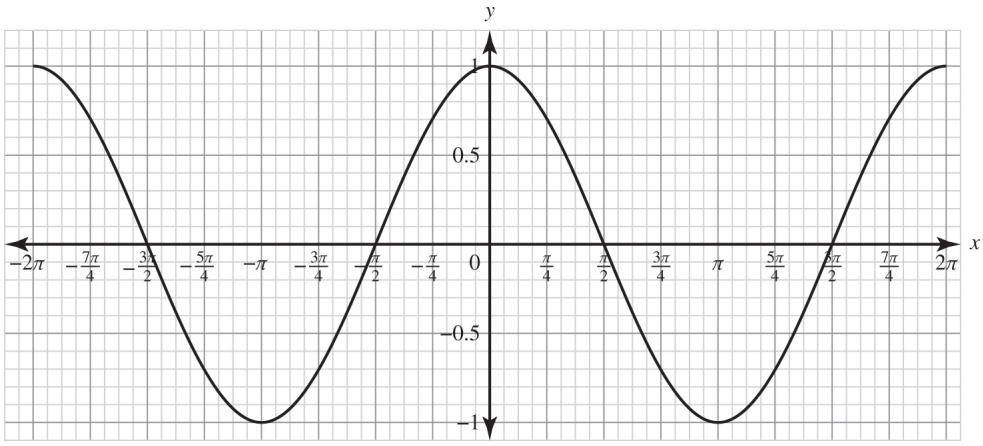
7DEOH

[		$-\frac{\pi}{2}$	9•	$-\frac{3\pi}{2}$	9 •
\					

3ORWWLQJ WKH SRLQWV)URP 7DEOH DQG WR WKH



)LJXUH

6LQXVRLGDO)XQFWLRQ7UDQVIRUPDWLRQV

)XQFWLRQV WKDW FDQ .EHDZUEQWFWHGRULQLV
IRUJP... DFREV.. F G ZKHDUEHF DQG UHDO QX
@ DUH VLQXVRKLGDUHXQFWLRQVRUPDWL
FRVLQH IXQFWLRQV WUDQVIRUPDWLRQV 6HFWLRQ
FDQ EH DSSOLHG WR WKH JUDSKV RI WKH

$PSOLWXGHDVRI[DQG DFR[V

7KH YDOXHV WKH DPSOLWXGH RI WKH
7KH DPSOLWXGH RI WKH FRVLQH IXQFWLRQ
RU... FR.V

([DPSOH

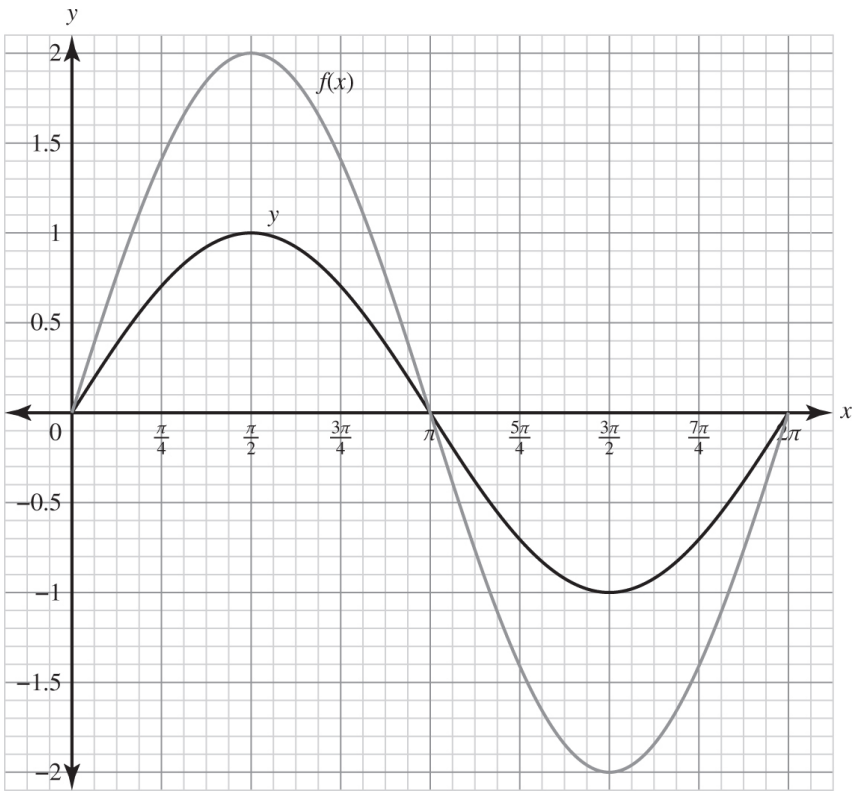
&RPSDUH HDFK RI WKH IROORZLQJ IXQFWL
IXQFWLRQV DQG OLVW WKHLU WUDQVIRU
VHW RI D[HV DV WKHLU SDUHQW IXQFWLF

I[VLQ

$$g(x) = \frac{1}{2}\cos x$$

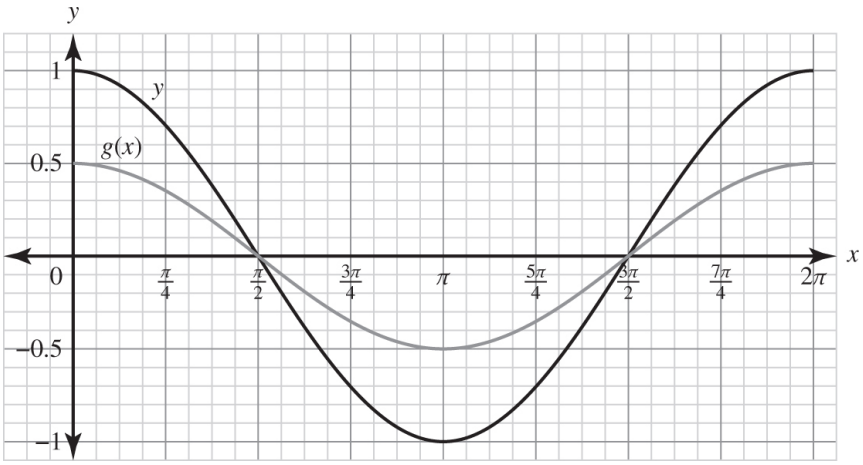
6RXWLRQ

7KH SDUHQW I[IXQFWLRQVLQ[KH JUDSKL
YHUWLFDO GLODWLRQ E\ D IDFWRU R
YDOXHV RI LWV SDUHQW IXQFWLRQ D
PLQLPXP YDOXHV RQ H DQG7KGHUHIRUH
DPSOL $\frac{2-(-2)}{2}=2$ L7KH JUDSKV RI WKH WZR
GRPDLQ > •@)DUHUVHKRZQ LQ



)LJXUH

7KH SDUHQW $\frac{1}{2}\cos x$: WLWVR QRRVIK H JUDSKLRI
YHUWLFDO GLOD\WLRQFEH\ WDKHDPWDRLPXRP
YDOXHV RI LWV SDUHQW IXQFWLRQ DU
YDOX $\frac{1}{2}\cos x$ D JHD $-\frac{1}{2}$G7KHUHIRUH WKH DP
$\frac{\frac{1}{2}-\left(-\frac{1}{2}\right)}{2}=\frac{1}{2}$ 7KH JUDSKV RI WKH WZR IXQFW
DUH VKRZQ LQ



)LJXUH

)UHTXHQF\ RVILEQ DQG FRVE[

7KH YDEXH FRV WKH IUHTXHQF\ IRU 7KHH YDOXH VWDWHV WKH QXPEHU RI IXOO F\F LQWHUYDO > •@

7KH JUDSKV RQ WKH PXOWLSOLFDWLYH \ IXQFWLRQV LQ D KRUL]RQWDO GLOODQGGWHGRV SHULRG E\ |D I,DFWHRUHRDO WKH SHULRG I

$\frac{2\pi}{|b|}$

7KH VDPH WUDQVIRUPDWLRQ RI WKH FR

([DPSOH

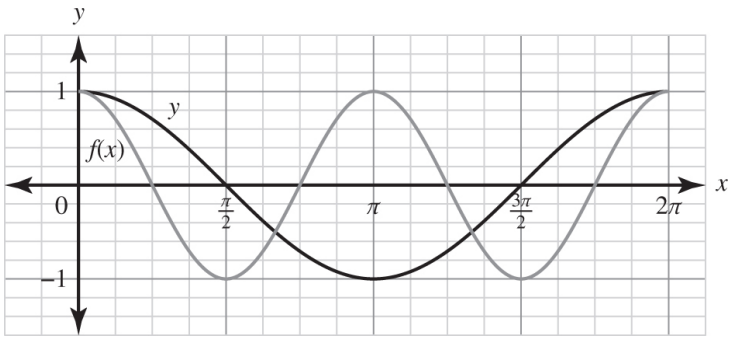
&RPSDUH HDFK RI WKH IROORZLQJ IXQFW IXQFWLRQV DQG OLVW WKHLU WUDQVIRU VHW RI D[HV DV WKHLU SDUHQW IXQFWLF

I[FRV

$g(x) = \cos\frac{1}{2}x$

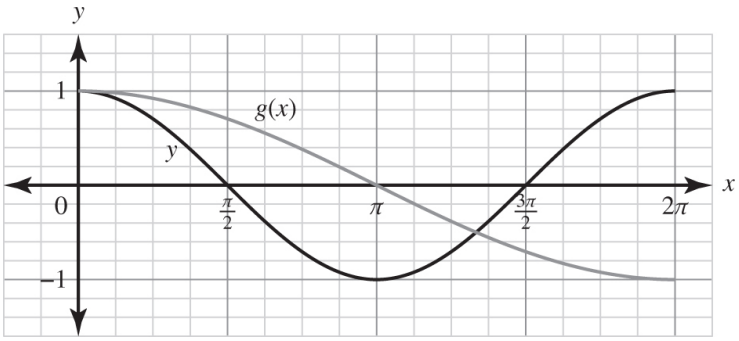
6ROXWLRQ

7KH SDUHQW I [IXQFWLRQ RI WKH JUDSK LV
KRUL]RQWDO GLODWLRQ RI IDFWRU E\ D
7KHUHIRUH WKH $PSOLWXGH RI WKH
I [FRPSOHWHV RQH FXOHJ EHKLQG•I
IXQFWLRQV RYHU WKH GRPDLQ >



)LJXUH

7KH SDUHQW g(x) = cos ½x IXQFWLRQ RI WKH JUDSK
KRUL]RQWDO GLODWLRQ RI IDFWRU E\ D
7KHUHIRUH WKH $PSOLWXGH RI 7KH SHULRG RI WKH [W V
FRPSOHWHV RQH KDOI RI 7KH FXOH EHKLQG•I
WZR IXQFWLRQV RYHU WKH GRPDLQ >



)LJXUH

9HUWLFDO 7UDQVODWLRQV DQG [,QWHUFHSWV

$GGLQJ D GRWWDKHWULJKW VLGH RI WKH I... V.LDQJG. F.V KLIWV WKH JUDSKV RIGW ! DQG GRZQ ZKHQPLGOLQH RI WKH HTXD HTXDWLRQ RI WKH PLGO\LQH LV WKH KRU

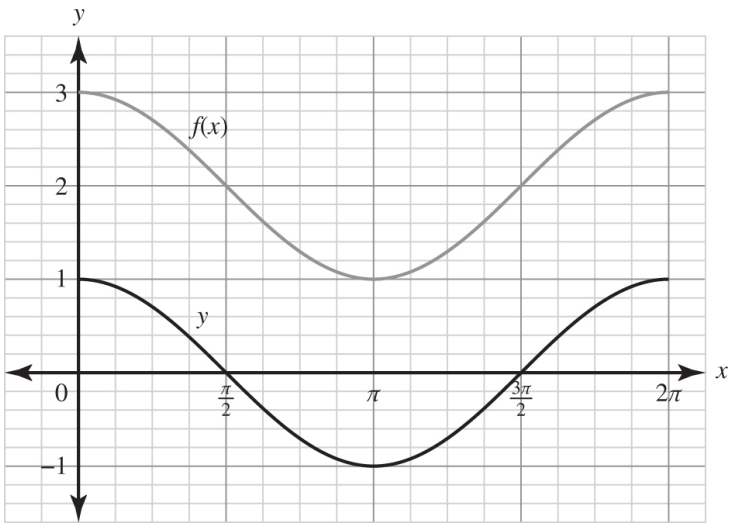
› ([DPSOH

&RPSDUH HDFK RI WKH IROORZLQJ IXQFW IXQFWLRQV DQG OLVW WKHLU WUDQVIRU VHW RI D[HV DV WKHLU SDUHQW IXQFWLF

I[F[RV
J[V[L9Q

✓ 6ROXWLRQ

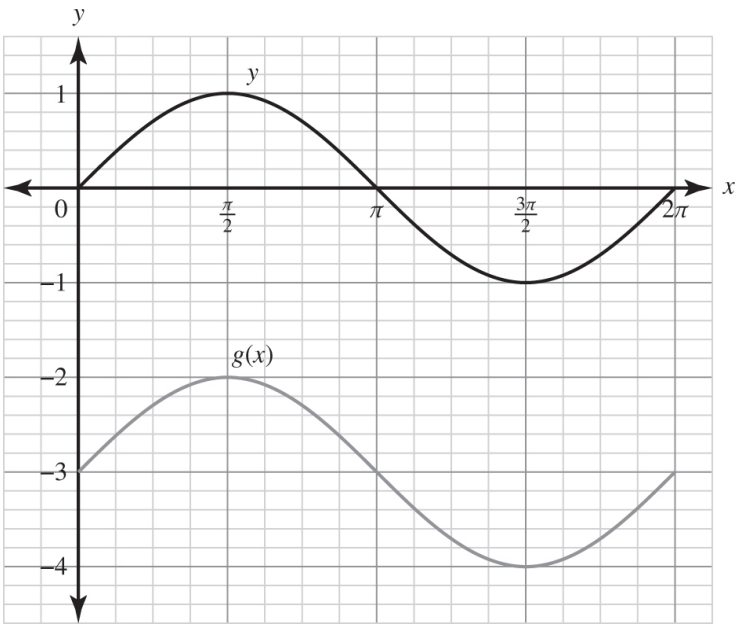
7KH SDUHQW I[IXQF[RWLRQVF[RV7KH JUDS LV D YHUWLFDO\SUDQVODWLRQ R \ 7KLV PHDQV WKDW WK[H LPD[LPXP PLQLPXP YD7KXHJUVSKV RI WKH WZR I GRPDLQ > •@)DUHK RZQ LQ



)LJXUH

7KH SDUHQW JUDSK RI WKH JUDSK RI D YHUWLFDO VWUHWFK DQG WUDQVODWLRQ R \ 9 7KLV PHDQV WKDW WKH PD[LPXP DQG PLQLPXP YDOXHV RI WKH JUDSK RI PXVW EH WZR XQLWV IURP WKH WZR I

> • @ DUH) IVJKXRUZH Q LQ



)LJXUH

+RUL]RQWDO VWUHWFKHV DQG FRPSUHVVLRQV DQG

$GGLQJ D FRQVWDQW WR WKH WULJRQRPHWULF IXQFWLRQ VKLIWV WKH JUDSK RI WKH IXQFWLRQ XS RU GRZQ

DQG ULJKW $ KRUL]RQWDO WUDQVODWLRQ LV FDOOHG D SKDVH VKLIW

([DPSOH

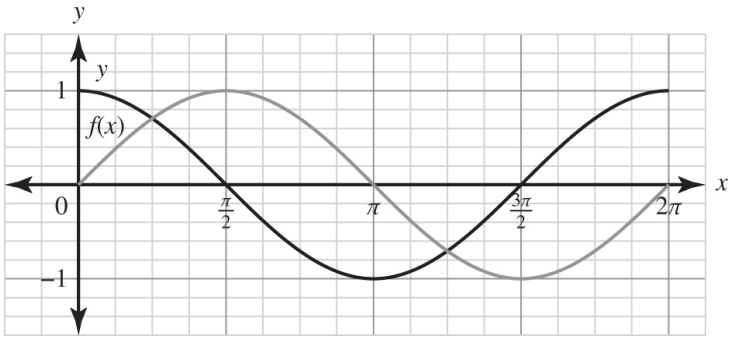
&RPSDUH HDFK RI WKH IROORZLQJ IXQFWLRQV DQG OLVW WKHLU WUDQVIRUPDWLRQV DV WKHLU SDUHQW IXQFWLRQ

$$f(x) = \cos\left(x - \frac{\pi}{2}\right)$$

J [V[L Q •

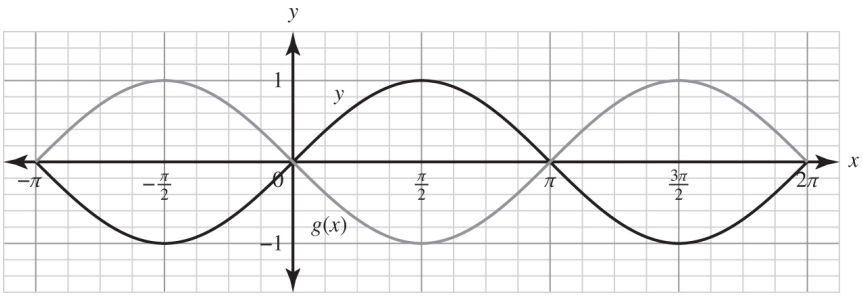
✓ 6 R O X W L R Q

7 K H S D U H Q W $f(x) = \cos\left(x - \frac{\pi}{2}\right)$ L E \ Q R [R 7 K H J U D S K L
D K R U L] R Q W D O W U D Q V O D W L R Q V R I W K H J U D S R I
I X Q F W L R Q V R Y H U W K H G R P D L Q • 7 K H •
J U D S $f(x) = \cos\left(x - \frac{\pi}{2}\right)$ F R L Q F L G H V Z L W K W K H J



) L J X U H

7 K H S D U H Q W J [I X Q F W L R Q • R Q L \ R V [L 7 K H J U D S K
L V D K R U L] R Q W D O W U D Q V O D W L R Q V R I W K H
I X Q F W L R Q V R Y H U W K H G R P D L Q • 9 • •

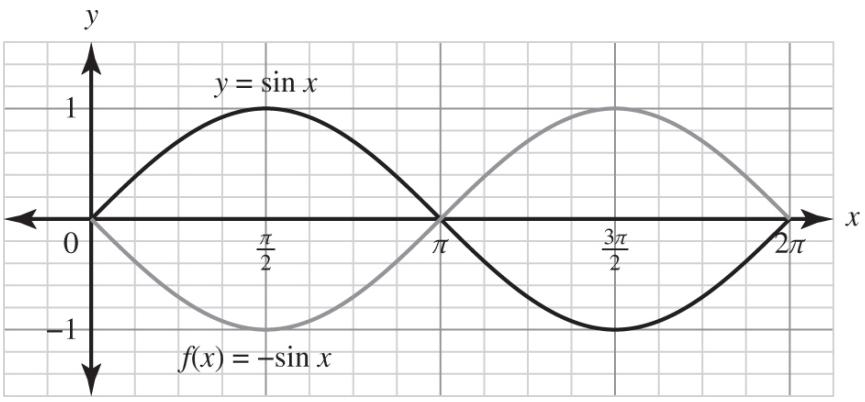


) L J X U H

5 H I O H F W L Q R 7 R P H W U L F) X Q F W L R Q V

7 K H J U D S K V R I V L Q H D Q G F R V [L Q H Q H Q V E O
V D P H Z D \ W K D W D O O S D U H Q W I X Q F W L R Q

7KH JUDSKV 9RIVDQQG 9 FRVUH WKH UHIC
JUDSKV RI WKH EDVLF VLQHDLQVG UFRVSDQVH
VKRZQ LQ)LJXUHV DQG



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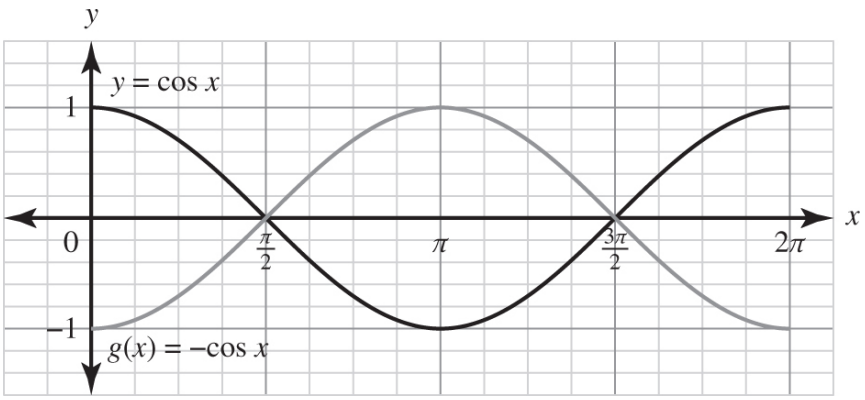
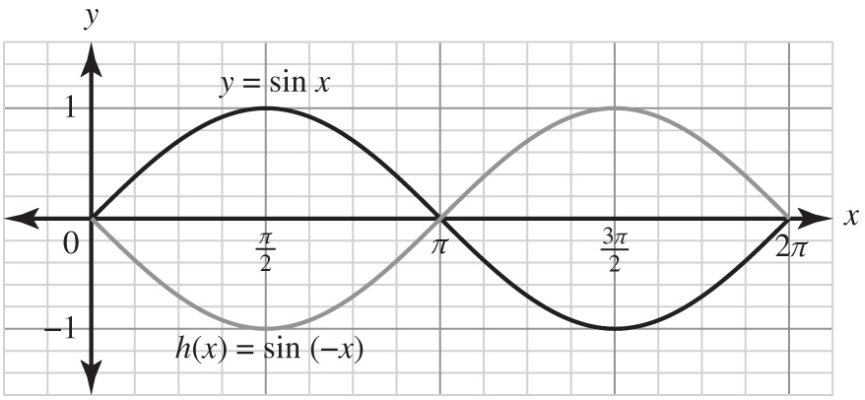
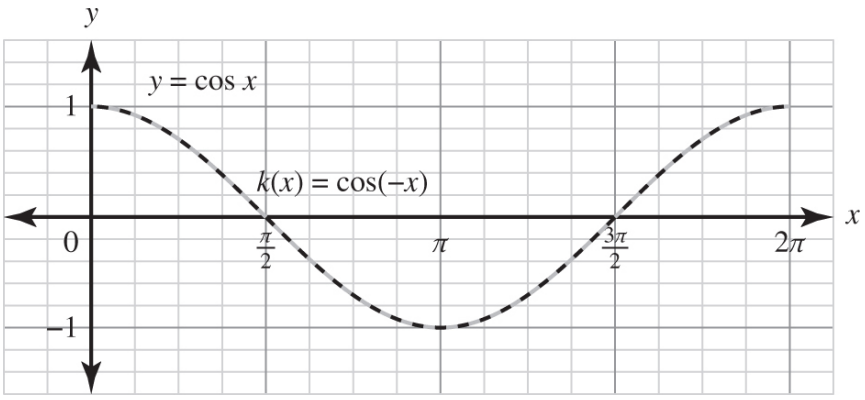


)LJXUH

7KH JUDKS[KV VRL[Q D9QG FRV UH WKH UH
WKH JUDSKV RI WKH EDVLF VDQLHVDQG FRV
)LJXUHV DQG 1RWH WKDW WKH FRVLQH FX
WR \WDK[HV VR WKH FRVSNGV RFR[V 9RLQFLG



)LJXUH



)LJXUH

$OO WKH WUDQVIRUPDWLRQV FDQ EH IRX
DQG FDQ EH JUDSKHG E\ KDQG RU XVLQJ GUDSK
JUDSKLQJ FDOFXODWRUV WR FKHFN VWHSV PDNH L
FRVLQH FXUYH E\ KDQG

*UDSKLQJ 6LQH DQG &RVLQH &XUYHV

67(3 8VLQJ WKH GHWHUPLQHG YDOXHV GHWHUPLQH WKH YDOXHV RI [WKDW PDNH \ PD[LPXP DQG PLQLPXP SRLQWV RI \

67(3 8VLQJ WKH GHWHUPLQHG YDOXHV ILQG $ WKH YHUWLFDO GLVWDQFH IURP RQH IXOO F\FOH RI WKH VLQH F

67(3 7R IILJXUH RXW K[RDZ[LVR GEDGHHWKH
TXDUSWHUL RG PDUNV SORW NH\ SRLQ
PLQLPXP DQG LQWHUFHSWV

67(3 8VLQJ FDOHX VKLIW WKH SRLQWV W
HLWKHU OHIW RU ULJKW

67(3 8VLQJ GDOHX VKLIW WKH SRLQWV W
HLWKHU GRZQZDUG XS JLYHV WKH HT
ZKLFK FDQ EH GUDZQ DV D KRUL]RQ

67(3 /RRN IRU QHJDWLRQV RI WKH[HRDD[
ZKHUH DSSURSULDWH

([DPSOH

*UDSK RQH IXOO F\FOH IRU HDFK RI WKH

I[V[LQ

$$g(x) = -2\cos\tfrac{1}{2}x$$

K[F[RV•

$$m(x) = \sin\left(2\pi x - \tfrac{\pi}{2}\right) - 1$$

6ROXWLRQ

)RU HDFK RI WKH JUDSKV IROORZ WKH
FRVLQH FXUYHV E\ KDQG

I[V[LQ

67(36

D

E VR $= \frac{2\pi}{2} = \pi$ LWRKGE YDOXH PHDQV WKH
F\FOHV IURP > •@ 6R WR HQVXUH
JUDSK ZLOO EH > •@

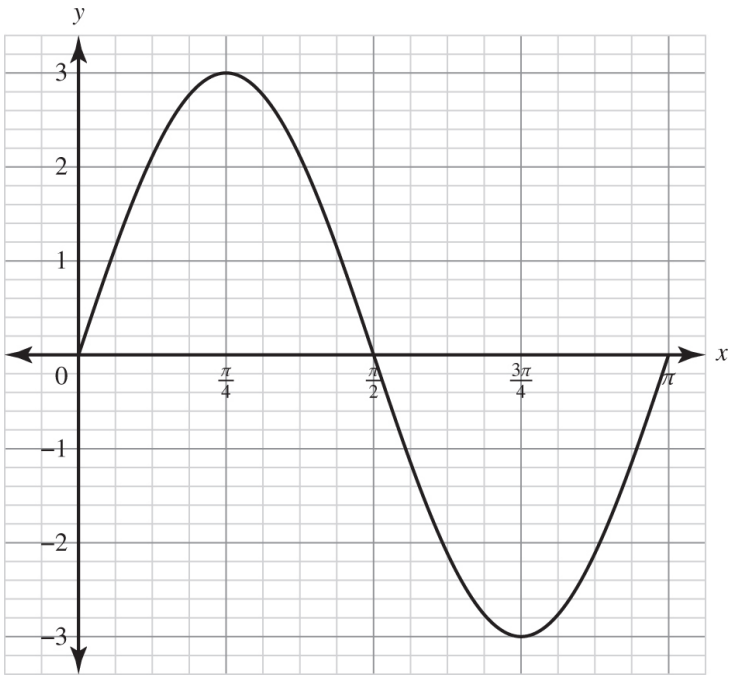
$$x\text{-scale} = \frac{\text{period}}{4} = \frac{\pi}{4}$$

F QR SKDVH VKLIW

G Q R Y H U W L F D O V K L I W

1 R Q H J D W L R Q V Q R U H I O H F W L R Q V

) L U V W V[FDD[O1 $\frac{\pi}{4}$ WDKQHG WKHQ EHQLWXKUDHVWDK
YDOXH RI DQG D PLQLPXP YDOXH RFXXHU RH
RULJLQ DQG WKHQ LQFUHDVHV GHFUH
F\FOH)OOORZ WKH RYHUDOO SDUHQW
SORWWHG RQHWKRGTXDWUDWLSKLVRIVKRZ(
) L J X U H



) L J X U H

$g(x) = -2\cos\frac{1}{2}x$

D 9 DPSOLWXGH _

$b = \frac{1}{2}$ VR S $= \frac{2\pi}{\frac{1}{2}} = 4\pi$ RTGKEH YDOXH PHDQV WKH
F\FOH IURP > •@ 6R WR HQVXUH
JUDSK ZLOO EH > •@

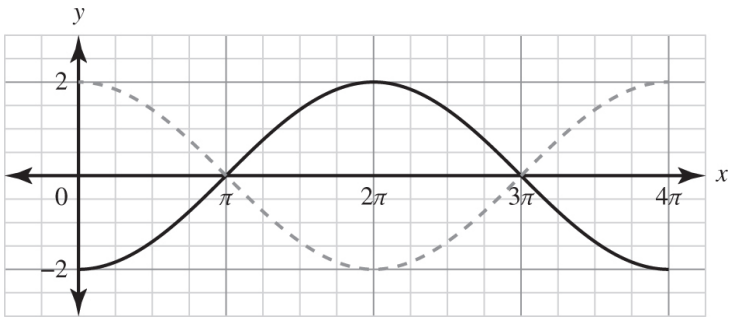
$x\text{-scale} = \frac{\text{period}}{4} = \frac{4\pi}{4} = \pi$

F Q R SKDVH VKLIW

G Q R Y H U W L F D O V K L I W

7KH RXWSXW LV QHJDWLYH DQG LVUHIOHFW

)LUVW VFDOH E\ WKH DQG WKHQ VKLIW GRZQ WR KDYH PLQLPXP YDOXH RI DQG D PLQLPXP YDOXH RI DQG PD[LPXP YDOXH DQG WKHQ GHFUHDVH F\FOH)ROORZ WKH RYHUDOO SDUHQW SORWWHG RQ WKH 7RROEDU 7KHQ VKLIW LW XS E\ XQLWV)LQDOO\ UHIOHFW LW RYHU [D[LV WR JHW WKH ILQDO VROXWLRQ



)LJXUH

K[F[RV•

D

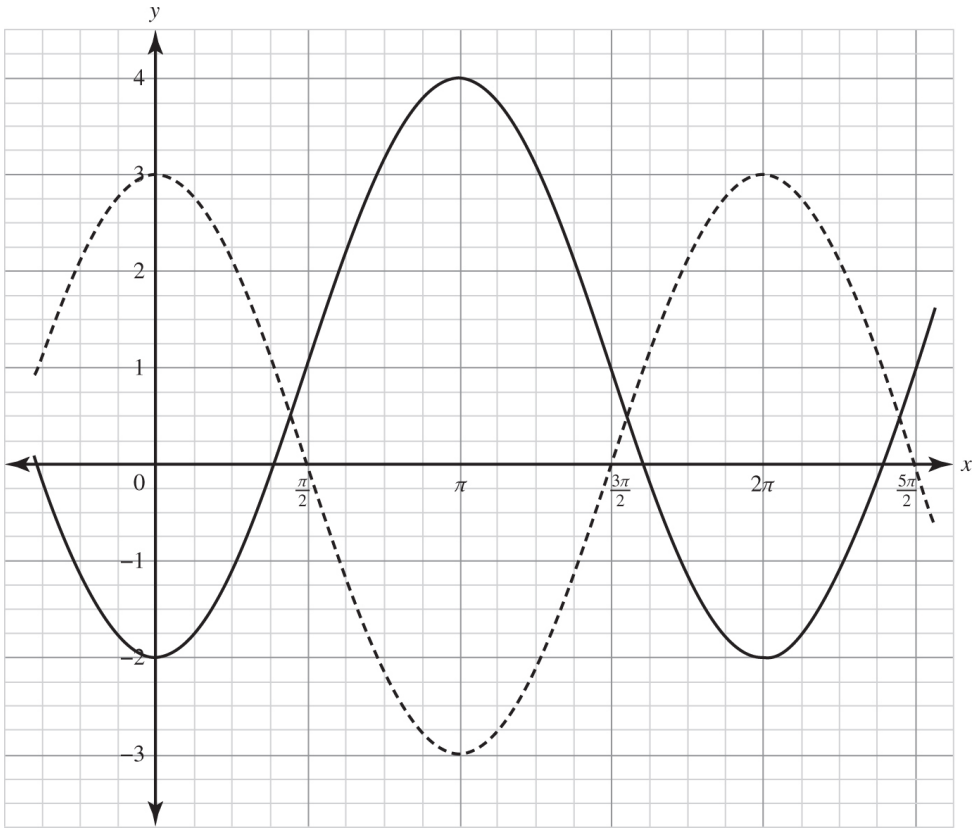
E VR $ 3HULRG $ YDOXH PHDQV WKH IURP > •@ 6R WR HQVXUH RQH IX ZLOO EH > •@

$ $ $

$ $ $

$ $ $

NH\ SRLQWV DUH SOSRHWWLRG7RQVWKHS
VKRZQ DV D GDVKHG7KLHQH WLQH WKH I
JUDSK DQG VKLIW WKHP OHIW • XQLW\
VKRZQ LQ UHDQG LV WKH ILQDO VROXW



)LJXUH

$$m(x) = \sin\left(2\pi x - \frac{\pi}{2}\right) - 1$$

D

7R ILQG YDOXHV WKDW PDNH WKLV HTXDWLRQ[ILUVW WKHR
SDUHQWKHVHV PXVW EH)DFWRU
SDUHQWKHVHV WR UHZULWH WKH HT

$$m(x) = \sin 2\pi\left(x - \frac{1}{4}\right) - 1$$

7KLV HTXDWLRQ ZLOO EH XVHG IRU
E • VR $= \frac{2\pi}{2\pi} = 1$ L7RKGE YDOXH PHDQV WKH
F\FOHV IURP > •@ 6R WR HQVXUH

JUDSK ZLOO EH > @

$x\text{-scale} = \frac{\text{period}}{4} = \frac{1}{4}$

$c = -\frac{1}{4}$ WKHUH LV D SKDVH VKLIW WR V

G 9 WKHUH LV D YHUWLFDO VKLIV

1R QHJDWLRQV QR UHIOHFWLRQV

)LUVW VFDOH WKH [-axis by $\frac{1}{4}$ EHFDXVH HQVXUH WKDW D PD

RI DQG D PLQLPXP YDOXH RI 9

DQG PLQLPXP YDOXHV DUH VHSDUDWHG E

WKH RULJLQ DQG WKHQ LQFUHDVHV G

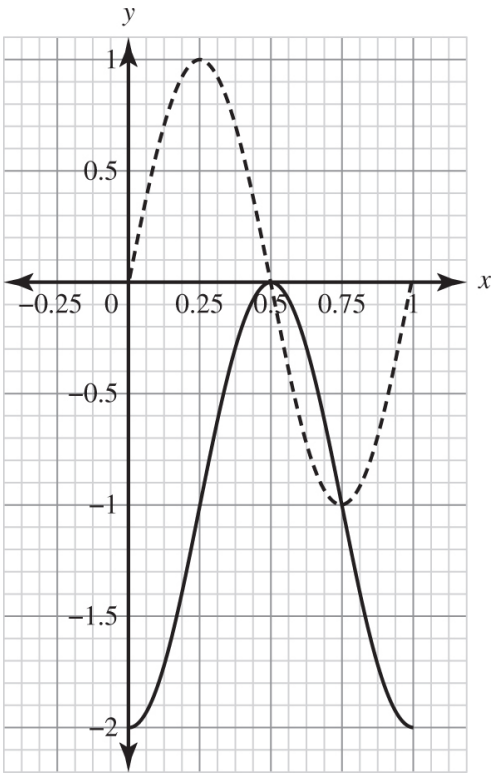
FRPSOHWH F\FOH)ROORZ WKH RYHU

NH\ SRLQWV DUH SORWWHG DQG WKHQ

VKRZQ DV D GDVKHG FXUYH 7KHQ

JUDSK DQG VKLIWHG XS XQLW

JUDSK VKRZQ LQ LV WKH ILQDO VR



)LJXUH

,W LV LPSRUWDQW WR QRW[HVWDKOWW ZLQVWE
QXPEHUV WKDW ZHUH QRW PXOWLSOHV F
ZKHQ WULJRQRPHWULF HTXDWLRQV DUH

› ([DPSOH

)RU HDFK RI WKH IROORZLQJ IXQFWLRQV
GHVFULEH WKH WUDQVIRUPDWLRQV WKDV

I[9 [V1L•

$$g(x)=\tfrac{1}{2}\cos\left(2x-\tfrac{\pi}{2}\right)-7$$

✓ 6RO XWLRQ

7KH SDUHQWI[XQF9WL[V1Q•IRU LV[L7KH JU
RI I[KDV D YHUWLFDO GLODWLRQ E\ D
WR WKH ULJKW • XQLWV LV YHUWLFDO
RYHU[DV[KLV

7KH SDUHQW$$g(x)=\tfrac{1}{2}\cos\left(2x-\tfrac{\pi}{2}\right)-7$$ LWRXUFR[R7R GHWH
WKH DSSURSULDWH WUDQVIRUPDWLRQ
WKH \RDUPEV 9F G)DFWRULQJ RXW IU
HTXDWLRQ FDQ EH UHZULWWHQ LQ WK

$$g(x)=\tfrac{1}{2}\cos 2\left(x-\tfrac{\pi}{4}\right)-7$$

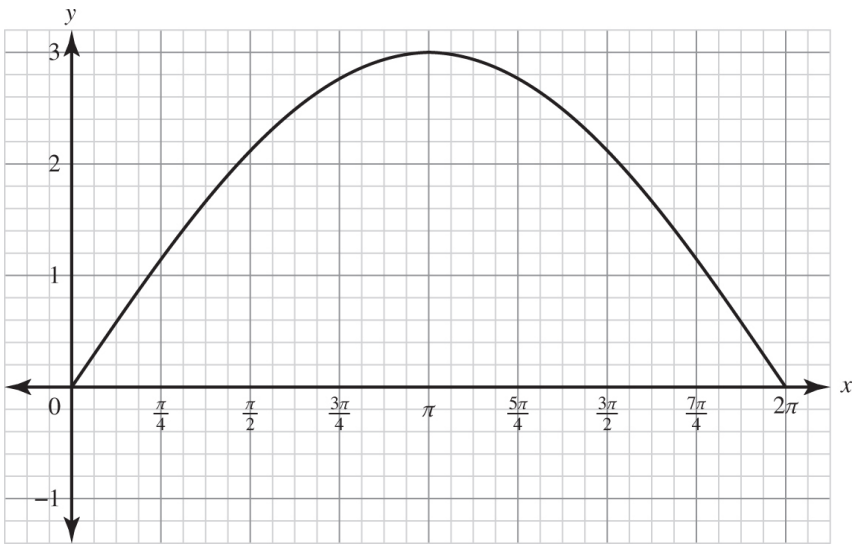
7KH JUDSKKRV D YHUWLFDO ($\frac{1}{2}$ LODKRUL
GLODWLRQ $\frac{1}{2}$ \ DVIRXWRJRWDOO\ $\frac{\pi}{4}$WXQLQ
DQG LV YHUWLFDOO\ WUDQVODWHG GR

6LQFH WKH JUDSKV RI WKH VLQH DQG F
EHFDXVH RI D SKDVH VKLIW ZKHQ JLYH
GDWD WKHUH DUH LQILQLWHO\ PDQ\ H
PRGHO IRU WKH GDWD (DFK VROXWLRQ

MXVW DQ H[DPSOH RI RQH SRVVLEOH H
WKH VPDOOHVW SKDVH VKLIW SRVVLEO

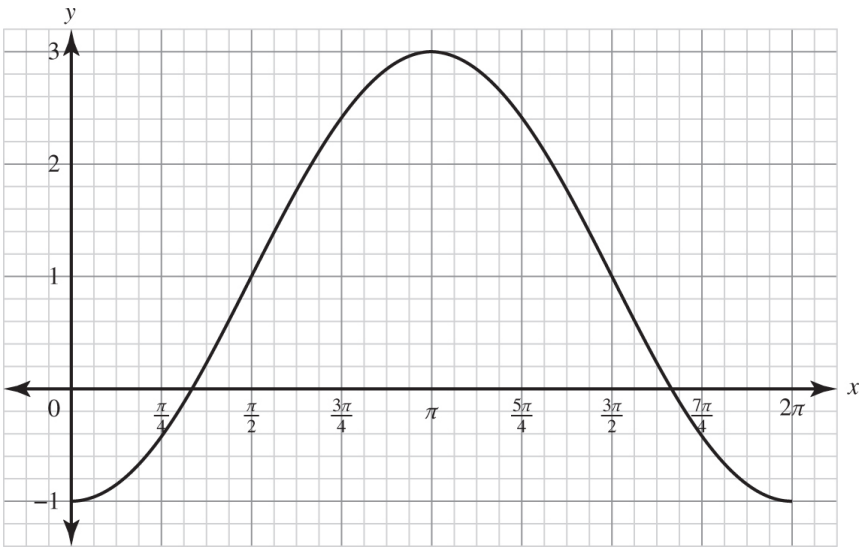
([DPSOH

:KDW LV DQ HTXDWLRQ RI WKH WULJXUH
"



)LJXUH

:KDW LV DQ HTXDWLRQ RI WKH WULJXUH
"



)LJXUH

✓ 6ROXWLRQ

)RU HDFK RI WKH HTXDWLRQV IROORZ W E\ KDQG

7KH FXUYH KDV WKH EDVLF VKDSH RI

7KH DPSOLWXGH LV WKH GLVWDQFH PLGOLQH D

7KHUH LV KDOI RI D IXOO F\FOH RI D $b = \frac{1}{2}$ 6R WKH $= \frac{2\pi}{\frac{1}{2}} = 4\pi$ ULRG

7KH $\text{HVF} = \frac{4\pi}{4} = \pi$ 7KH NH\ SRLQWV RI KDOI DUH RFFXUULQJ DW HYHU\ PXOWLSC

7KH VLQH FXUYH EHJLQV DW WKHFRU

7KH VLQH FXUYH EHJLQV DW WKH RU

7KH VLQH FXUYH EHJLQV DW WKH RU QR UHIOHFWLRQV WKHUH DUH QR Q

$Q HTXDWLRQ $y = 3 \sin \frac{1}{2}x$ KH FXUYH LV

7KH FXUYH KDV WKH EDVLF VKDSH RI

7KH DPSOLWXGH LV KDOI WKH GLVW
DQG PLQLP $= \frac{3-(-1)}{2} = \frac{4}{2} = 2 = a$

7KHUH LV RQH IXOO F\FOH RI D FRE
6R WK $= \frac{2\pi}{1} = 2\pi$ UL RG

7K[HVF $= \frac{2\pi}{4} = \frac{\pi}{2}$ 7KH NH\ SRLQWV RI D IXC
RFFXUULQJ DW $\frac{\pi}{2}$ YHU\ PXOWLSOH RI

7KH JLYHQ FXUYH EHJLQV DW WKH P
EHJLQV DW WKH PD[LPXP YDOXH WK

7KH PLGOLQH LV WKH DYHUDJH RI W
$= \frac{3+(-1)}{2} = \frac{2}{2} = 1 = d$

7KH FRVLQH FXUYH EHJLQV DW WKH
WKH JLYHQ FXUYH 7KHUHIRUH WKH
D[LV VDPH DV WKH H LV QHJDWHG

$Q HTXDWLRQ RI WKH FRVXUYH LV

6LQXVRLGDO)XQFWLRQ &RQVW[WDQG 'DWD 0RGHOLQJ

$V LQ WKH SUHYLRXV FKDSWHUV WULJR
GDWD 3KHQRPHQD WKDW DUH F\FOLF LQ
IXQFW7LKRQWHFKQLTXHV GLVFXVVHG LQ W
WR GHWHUPLQH RU HVWLPDWH HUURUV DQG
WR ZULWH WKH HTXDWLRQ RI WKH VLQXV

([DPSOH

7KH PHDQ PRQWKO\ PD[LPXP WHPSHUDW
7DEOH

7DEOH

0 R Q V	- D Q H E D U S S U O L D \ - X Q H K O \ X J 6 H S W W R Y H F
7H P S p &	

: U L W H D Q H T X D W L R Q W R P R G H O W K L V G

✓ 6 R O X W L R Q

6 L Q F H W H P S H U D W X U H L V D S H U L R G L F S K J L Y H Q G D W D 8 V L Q J W K H H T X D W L R Q W K H L R Q V W H S S U H Y L R X V V H F W L R Q Z L O O E H X V H G W R G

7 K $\text{amplitude} = \frac{\text{maximum} - \text{minimum}}{2} = \frac{28 - 15}{2} = 6.5 = a$

7 K H S H U L R G L V P R Q W K V V L Q F H W K H D I W H U H D F K Q F H $\frac{2\pi}{b}$ H U W $12 = \frac{2\pi}{b}$ Q D Q $b = \frac{\pi}{6}$ R

7 K H [V F D O H L V Q R W Q H F H V V D U \ K H U H W J U D S K H [G D [L W K Z R X O G X V H H D F K P R Q W K

7 K H Y H U W L F D O V K L I W V K R X O G E H I R X V K L I W H P L $= \frac{\text{maximum} + \text{minimum}}{2} = \frac{28 + 15}{2} = 21.5 = d.$

7 K H P R G H O $y = 6.5 \sin \frac{\pi}{6}(x - c) + 21.5$ 7 K H V L Q H F X U Y H P L G O L Q H Z K L F K L Q W H U V H F W V W K H D [L V D W W K D W R I 2 F W R E H U P D [L P X P W H P S H 7 U K D L W X F D Q Z E D V W K H S R L Q W V L V S R L Q W F D Q E H X V H G V L Q H F X U Y H P H D Q L Q J W K D W W K H F X U 7 K H U H F R U H

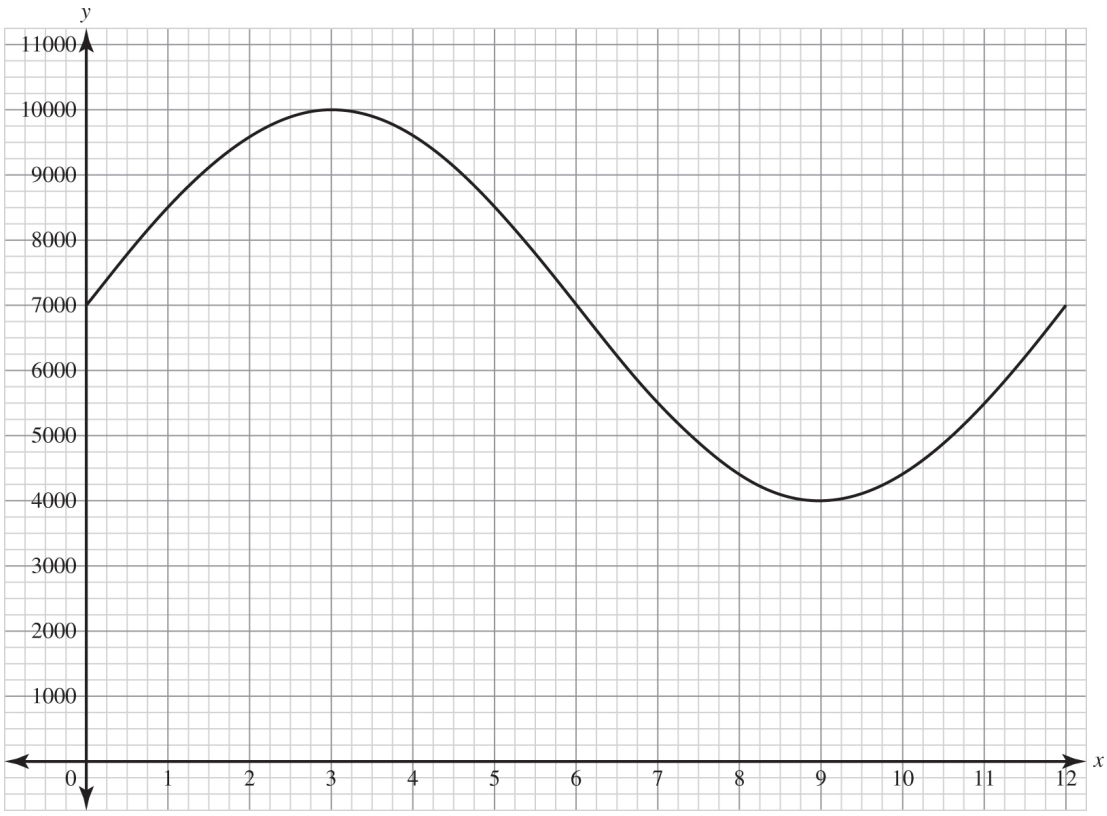
$ I W H U W K H S K D V H V K L I W W K H F X U Y H I R O O R Z L Q J W K H E H K D Y L R U D O H Q H U H I O H F W L R Q V

7 K H H T $y = 6.5 \sin \frac{\pi}{6}(x - 10) + 21.5$ L V W K H P R G H O

› ([D P S O H

7 K H Q X P E H U R I Y L V L W R U V D W D U H V R U W W R W K H L H Q W H U P L Q H D Q H T X D W L R

WKH PRQWK QXPEHU



)LJXUH

☑ 6RO XWLRQ

6LQFH WKH FXUYH KDV WKH EDVLF VKDSH IRU\P DVLQ F G7KH VWHSV IURP WKH SU XVHG WR GHWHUPLQH

7 K$$\text{amplitude} = \frac{\text{maximum} - \text{minimum}}{2} = \frac{10{,}000 - 4{,}000}{2} = 3{,}000 = a$$

7KH SHULRG LV PRQWKV VLQFH WKD JUDSK WR JHW RQH I$$\text{period} = \frac{2\pi}{b} \text{ FWO } 12 = \frac{2\pi}{b} \text{ ID VQLG}$$

$$b = \frac{\pi}{6}$$

7K+HVFDOH LV QRW QHFHVVDU\ KHUH W

7KH YHUWLFDO VKLIW VKRXOG EH 7RR

$$\text{midline} = \frac{\text{maximum} + \text{minimum}}{2} = \frac{10{,}000 + 4{,}000}{2} = 7{,}000 = d$$

7KH PRGHO KDV QR SKDVH VKLIW

7KH FXUYH EHJLQV DW WKH PLGOLQH RI D JHQHUDO7KWHLUQHHDFXHUQHR UHIOHFWL

7KH HT $y = 3{,}000 \sin\left(\frac{\pi}{6}t\right) + 7{,}000$ LV WKH PRGHO

7HFKQRORJ\ FDQ EH XVHG WR ILQG DQ PRGHO IRU D GDWD VHW

([DPSOH

7DEOH VKRZV WKH PHDQ PRQWKO\ PD[LPX %HWKSDJHRUN1HZQ WKH \HDU

7DEOH

0RQV	-DQHEDUSUDO\-XQ-XOXJ6HSW2FW1RYHF
7HPS p&	

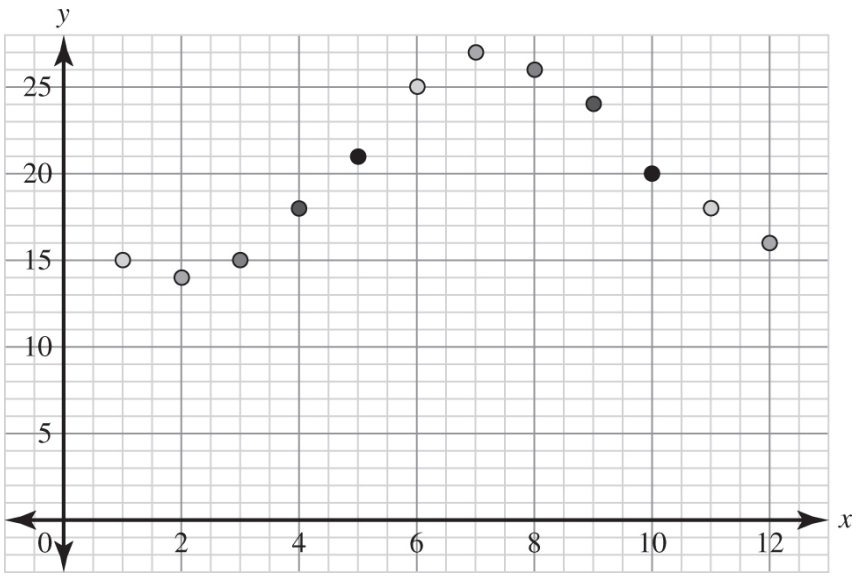
D 8VLQJ -DQXDU\ )HEUXDU\ HW7F DVLEQ 9F GFDQ EH XVHG WR PRGHO WK ZLWKRXW XVLQJ WHFKQRORJ\

E 8VH WHFKQRORJ\ WR FKHFN \RXU DQV PRGHO ILW"

F 8VLQJ \RXU PRGHO IURP SDUW E DS ZDV LQ 2OG %HWKSDJH LQ)HEUXDU\ UHDVRQDEOHQHVV RI WKLV DSSUR[LP

6ROXWLRQ

D 6LQFH WKH HTXDWLRQ9LDRQLEQ9F WGKWKHRUPV IURP WKH SUHYLRXV VHFWLRQ DORQJ XVHG WR GHWHUPLQH VKRZV WKH VI



)LJXUH

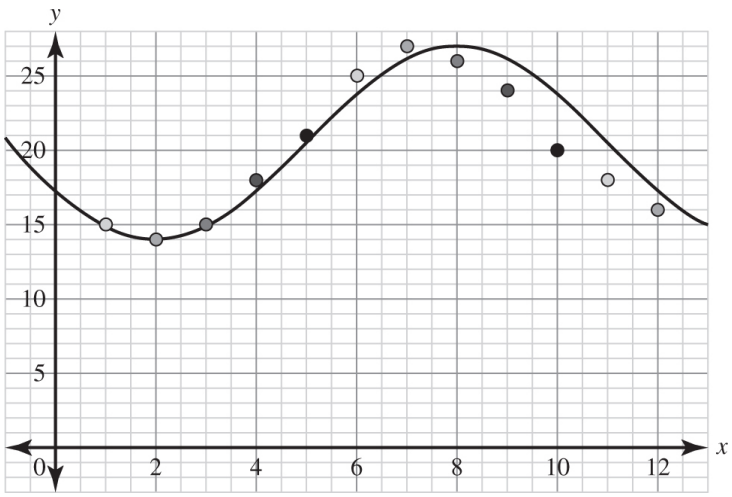
7K$PSOLWXGH $PSOLWXGH $PSOLWXGH $

WKH PRQWKV QXPHULFDOO\ LQWR / D
/ $IWHU WKH GDWD KDYH
6HOHFW &7 K
ZDQWHG ,Q WKLV FDVH HQWHU (Q
&DOFXODWH WR [LP DWLQJ WKH
WKH HTXDWLRQ LQ P 9

7R FRPSDUH WKLV HTXDWLRQ WR WKH
YDOXH IURP WKH LQSXW

$$T = 6.129 \sin 0.597(m - 4.757) + 20.557$$

7KH DQG YDOXHV DUH FORVH WR WKH Y
YDOXH IRXQG $ UZK LFDK ZDV FORVH
YDOXH R7KH SKDVH VKLIW E\ DTKXHH VDGLI
HTXDWLRQ RSWHG WR VKLIW WKH IXQ
WKH GDWD LQ WKH WDEOH WKH WHPS
EHWZHHQ p& DQG p& ZKLFK IDOO L
6R WKLV LV D UHDVRQDEOH DSSUR[LPDWLRQ
JUDSKHG DORQJ ZLWK WKH DVVRFLDWHG
FDOFXODWHG E\ WKH FDOFXODWRU LV
)LJXUH %RWK VHHP WR DSSUR[LPDWH
WKH HTXDWLRQ IRXQG XVLQJ WHFKQR
GDWD SRLQWV



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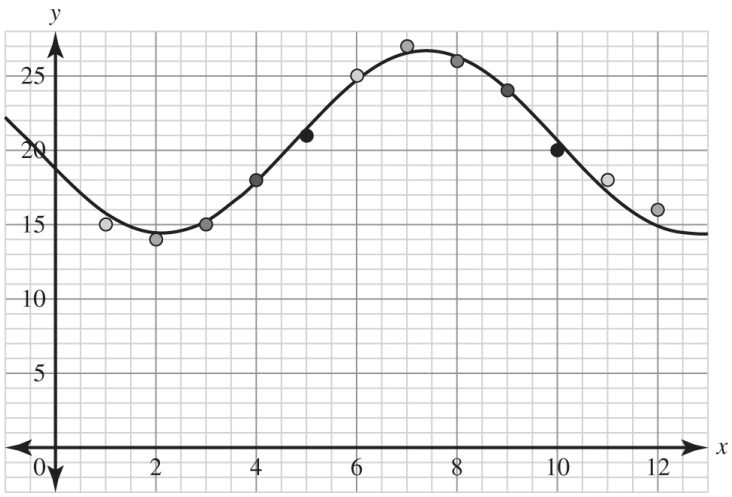


)LJXUH

F 7R DSSUR[LPDWH ZKDW WKH WHPSHUD
)HEUXDU\ VXEVHTXHQWO\ WKH WHP
P 9 DQG 7 VHFRQGV WKH UHDW
FDOFXODWRU LV VHW WR UDGLDQ PRC

$$T = 6.129 \sin 0.597((14) - 4.757) + 20.557 \approx 16.3119°\text{C}$$

:KHQ FRPSDUHG ZLWK WKH 7DEEOUHX
WHPSHUDWXUH LV RVFLOLQRUN
VKRZV WKDW WKH HTXDWLRQ ZDV
H[SODLQLQJ WKH SRVVLEOH RYHUHVW

([DPSOH

7KH KHLJKW WHUV RI WKH WLGH DERYH
%HDFK LV PRGHOHG $ KHWUDWLRQ X
KRXUV DIWHU PLGQLJKW

D *UDSK IRUWB B

E :KHQ ZDV KLJK WLGH DQG ZKDW ZDV

F :KDW ZDV WKH KHLJKW DW

G ,I D VKLS FDQ FURVV WKH KDUERU SU
DERYH PHDQ VHD OHYHO ZKHQ LV FU

☑ 6ROXWLRQ

D 7R JUDSK IROORZ WKH VWHSV DQG I
WR UDGLDQ PRGH

6LQDF H WKH DPS7KLHWXGH[LPXP YDOX
PLQLPXP YDOXH LV

6LQ$b = \frac{\pi}{6}+ Wperiod = \frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{6}} = 12$

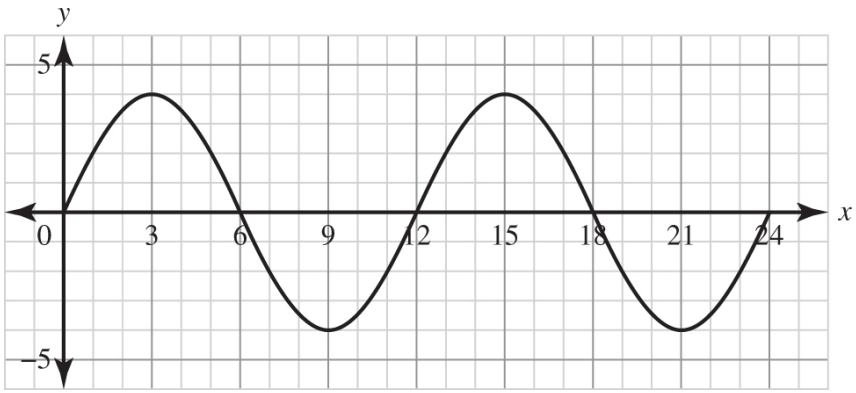
$x\text{-scale} = \frac{\text{period}}{4} = \frac{12}{4} = 3$

6LQF H WKHUH LV QR SKDVH VKLIW

6LQG H WKHUH LV QR YHUWLFDO V

6LQF H WKHUH DUH QR QHJDWLRQV

7KH JUDSK L)V JVKRZQ LQ



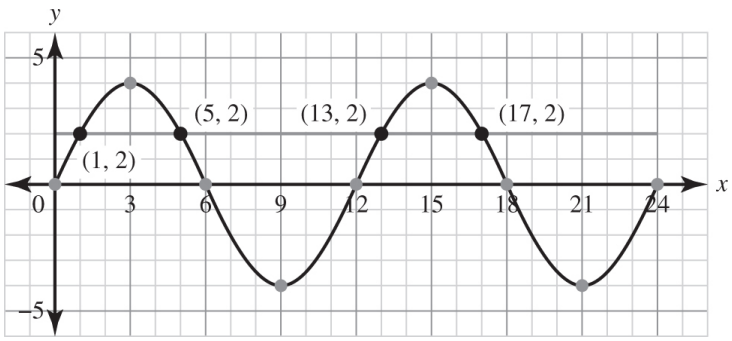
)LJXUH

E +LJK WLWH LDWQG W ZKLF$K 0DUHG 7KH
PD[LPXP KHLJKW LV PHWHUV DERYH

F $W30 W 7R ILQG WKH K$h(14) = 4 \sin\left(\frac{\pi(14)}{6}\right) \approx 3.464$ X
PHWHUV 7KUH WLGH LV PHWHUV DERYH

G 7R ILQG WKH WLPH ZKHQ IWR VDLQG LW
7R ILQG WKH VROXWLRQ XV$Y1 = 4 \sin\left(\frac{\pi t}{6}\right)$ JUDQ

< 7KHQ ILQG WKHLU SRLQWV)RJXUH



)LJXUH

)URP WKH JUDSK WKHUH DUH IRXU SRLQWV DQG 6LQFH LW ZDV JLYHQ WKDW ZKHQ WKH WLGH LV DW OHDVW PHWHUV DERYH WKH DYHUDJH OHYHO

7KH 7DQJHQW)XQFWLRQ

7KHUH DUH RWKHU WULJRQRPHWULF IXQFWLRQV LQFOXGLQJ6HFDQW&RVHFDQW&RWDQJHQW DQG&RWDQJHQW UHODWLRQVKLS ZDV GHWHUPLQHG EHWZHHQ DQG FRVLQH RI WKDW DQJOH

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$

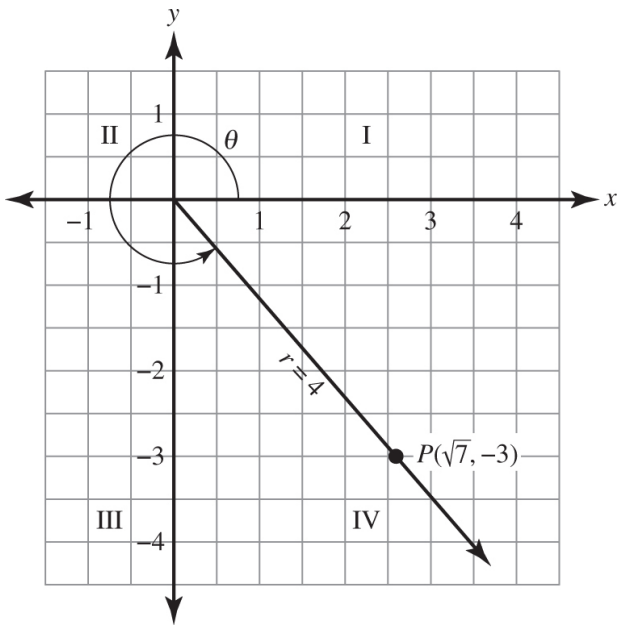
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cos\theta \neq 0$$

([DPSOH

, $P(\sqrt{7}, -3)$ LV D SRLQW RQ WKH WHUPLQDO VLGH RI DQ DQJOH

6RROXWLRQ

7KH JLYHQ LQIRUPDWLRQ)LJXUH DSKHG DQ



)LJXUH

8VLQJ WKH FRRUGLQDWHV RI 3 DQG WKH IDFW WKDW $ LV RQ WKH WHUPLQDO VLGH RI WKLV SRLQW OLHV RQ WKH FLUFOH RI UDGLXV FHQWHUHG DW WKH RULJLQ D

$$x^2 + y^2 = r^2$$
$$(\sqrt{7})^2 + (-3)^2 = r^2$$
$$16 = r^2$$

6LQFH UHSUHVHQWV D GLVWDQFH LW LV D

$$r = \sqrt{16} = 4$$

3RLQW 3 GRHV QRW OLH RQ WKH WHUPLQDO VLGH RI DQ DQJOH RI GHJUHHV RU UDGLDQV VR WKH DQJOH FDQ VWLOO EH GHWHUPLQHG

$$\sin\theta = \frac{y}{r} = \frac{-3}{4}$$
$$\cos\theta = \frac{x}{r} = \frac{\sqrt{7}}{4}$$
$$\tan\theta = \frac{y}{x} = \frac{-3}{\sqrt{7}} = \frac{-3\sqrt{7}}{7}$$

*UDSKIRI WDQ

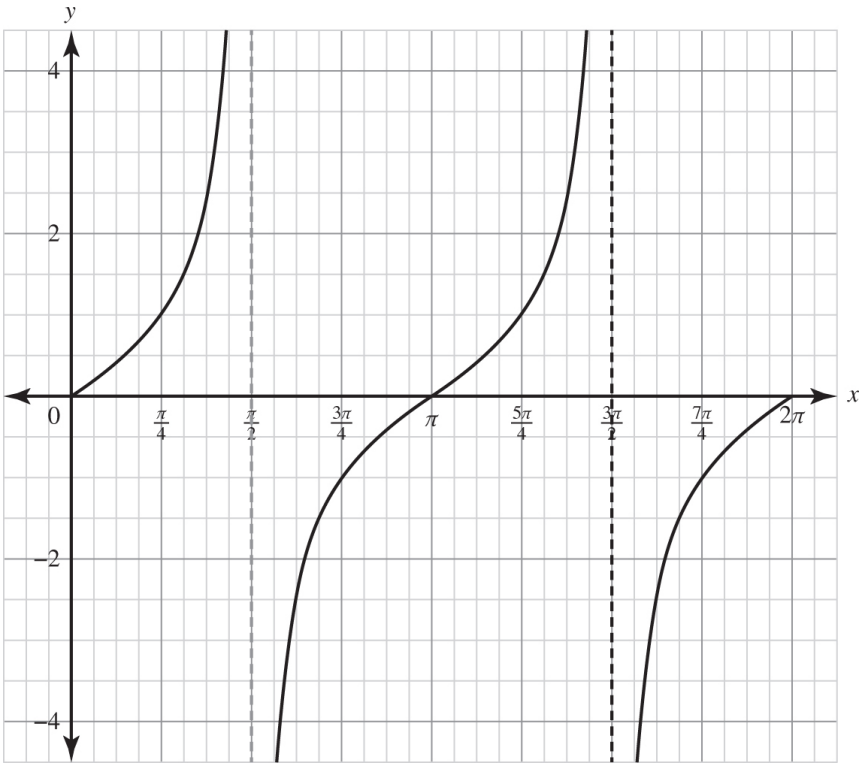
7KH JUDSK RI WKH WDQJHQW FXUYH DQG FRV WKH VLQH DQG FRVLQH IXQFWLRQV F WDNH WKH UDWLR RI WKH VLQH DQG FRV

7DEOH

7DEOH

		$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	•	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	•
VLQ		$\frac{1}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$		$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$		$-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$		$-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$	
FRV		$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{1}{2}$		$-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$		$-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$		$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	
WD	$\frac{0}{1}=0$	$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	$\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}=1$	$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}$	$\frac{1}{0}$ XQGH	$\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}=-1$	$\frac{0}{-1}=0$ QHC	$\frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}=1$	$\frac{-1}{0}$ XQGH	$\frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}=-1$	$\frac{0}{1}=0$ QHG

)UR 7DEOH WKH IXQFWLRQV QRW $\theta=\frac{\pi}{2}$ RQH GG RGG LQWHJHU $\frac{\pi}{2}$ WKH YHUWLFDO DV YLVXDOL



)LJXUH

7KH *UDSK RI WKH)XQFWLRQ

/LVWHG EHORZ DUH NH\ IDFWV DERXW W
IXQFWLRQ

- 7KH SHULRG RI WDQJH
  IUF$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ 7KH OHQJWK RI WKLV LQWHU
- 7KHUH LV QR DPSOLWXGH VLQFH WK
  PD[LPXP RU DEVROXWH PLQLPXP YD
- 7KH GRPDLQ LV DOO UHDO QXPEHUV
  $\pm\frac{\pi}{2}$
- 7KH UDQJH LV DOO UHDO QXPEHUV
- 7KH [LQWHUFHSWV RFFXU•DW LQWH
- 7KH YHUWLFDO DV\PSWRWHV RFFXU
  $\pm\frac{\pi}{2}$ 7KLV PHDQV WKDW WKH WDQJHQW
  SHULRGLF DV\PSWRWLF EH$\theta = \frac{\pi}{2} + k\pi$ Y LRU D

LQWHJHU YDOXHJUDSK FRPSOHWHV
EHWZHHQ FRQVHFXWLYH YHUWLFDO
- 7KH WDQJHQW IXQFWLRQ LQFUHDVH
- 7KH JUDSK FKDQJHV IURP FRQFDYH
EHWZHHQ FRQVHFXW7LKYHH SDRVLQWWRRIWL
H[LVW[VLQWHUFHSWV

7UDQVIRUPDWLRQWDRI

7KH WUDQVIRUPDWLRQV WKDW ZHUH DSS
FDQ DOVR EH..DSWGZQHGK WRKH VDPH UHVX

$GGLWLYH7IRUPDWLRQV

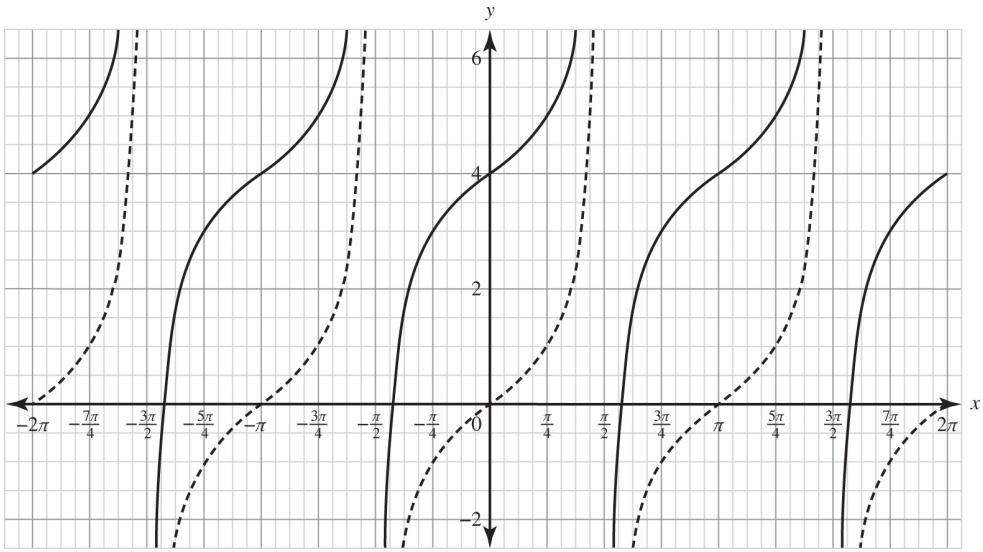
7KH JUDSK RI WKH DGGLWLYHGWUDQVIRUP
IXQFWLRQ D YHUWLFDO WUDQVODWLRQ
7KH JUDSK RI WKH DGGLWLYHRUDQVIRUP
IXQFWLRQ D KRUL]RQWDO WUDQVODW
RI EFXQLWV

([DPSOH

'HVFULEH WKH WUDQVIRUPDWLRQV.RQ ZKH
FRPSDUHG ZLWK WK.H SDUHQWDSKXHQFWLRQ
ZLWK LWV SDUHQW IXQFWLRQ RQ WKH VD

6RXWLRQ

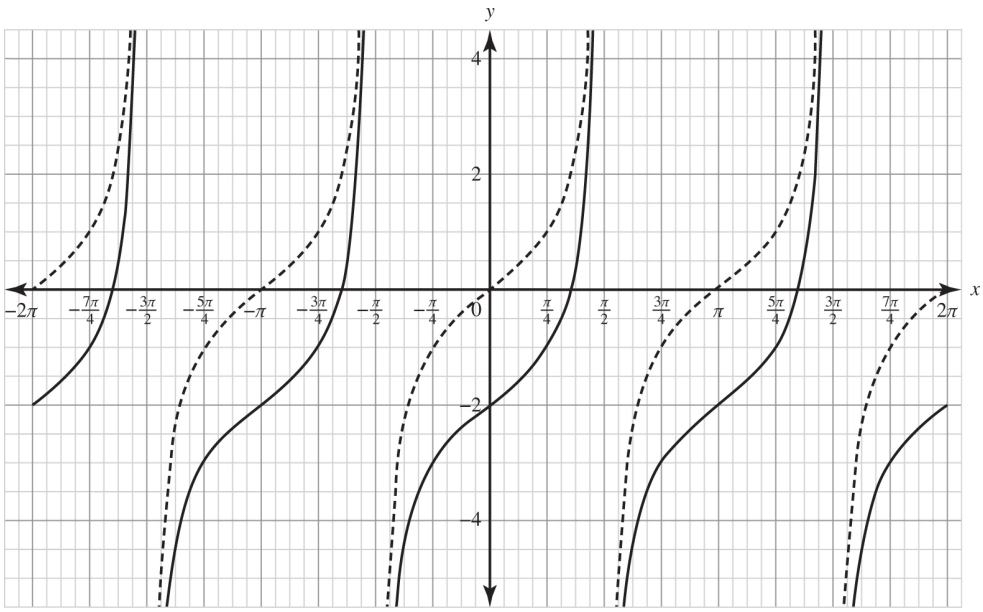
7KH JUDSK RI.DQ LV D YHUWLFDO VKLIW
I… W.DWKH JUDSKRVROG DQG.WKHGUDZ
)LJXUH



)LJXUH

7KH JUDSK RI I ... LV D YHUWLFDO VKLIW ... WKH JUDSK RI I ... WKDW KDV VRPH DGGLWLRQDO WUDQVIRUPDWLRQV

)LJXUH



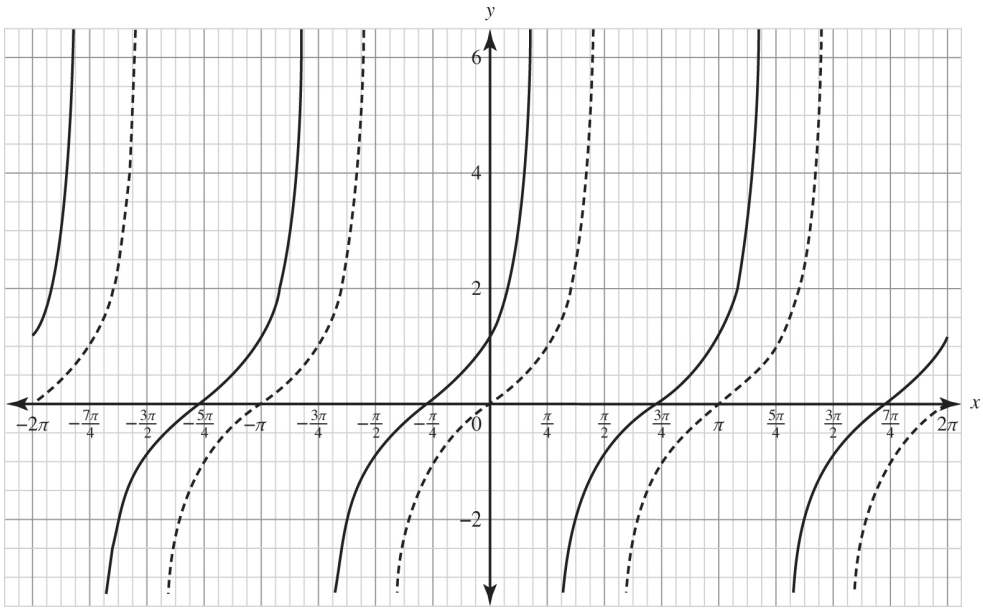
)LJXUH

› ([DPSOH

'HVFULEH WKH WJU.DQWW.RQPDKQLGRQWR9Q Z
FRPSDUHG ZLWK WK.H SW.DKQDSKXHQFWLRQ
ZLWK LWV SDUHQW IXQFWLRQ RQ WKH VD

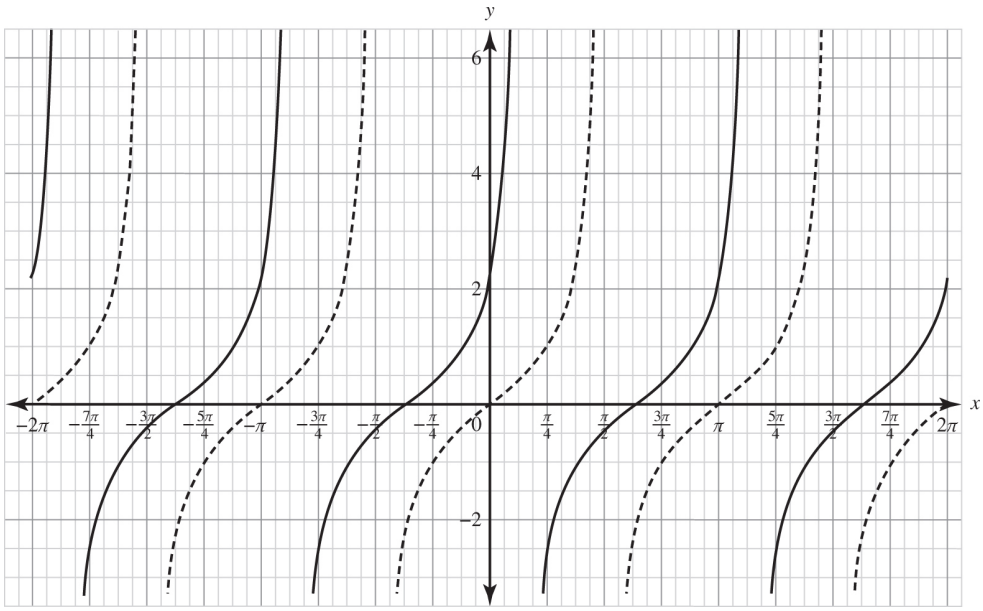
✓ 6ROXWLRQ

7KH JUDS.K RW.DQ LV D KRUL]RQWDO VKLI
RI … W.DWKH JUD.SK.VRIVROLG DQG. WVKHGDV
)LJXUH



)LJXUH

7KH JUDKS.K RW.DQ LV D KRUL]RQWDO VKL
RI … W.DWKH JUKD.SK.VRIVROLG DQG. WVKHGDV
)LJXUH



)LJXUH

0XOWLSOLFDQWWLRHU P7DWLRQV

7KH JUDSK RI WKH PXOWLJS.OLFDWDQLYWKWHU IXQFWLRQW.LRQW.DLQV D YHUWLFDO GLOEDWLRQFDW_

,ID WKH WUDQVIRUPDWLRQ [LQY[RVOYH

7KH JUDSK RI WKH PXOWLKS.OLFWEDQRY HVKW WDQJHQW..IXQW.DLQVRQ KRUL]RQWDO GLDQGHUV LQ SHULRG E\ D IDFWRU RI

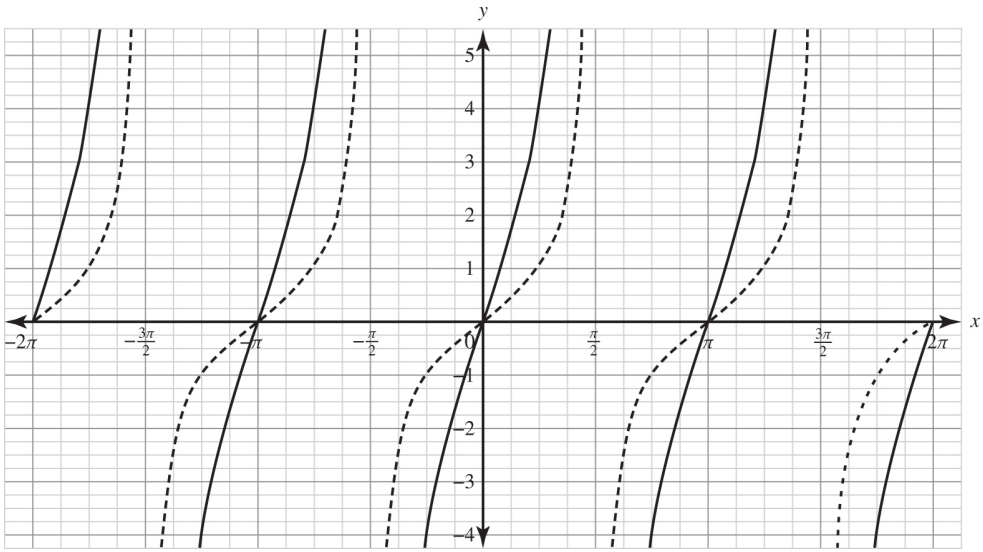
,IE WKH WUDQVIRUPDWLRQ L.QY[RVOYH

([DPSOH

'HVFULEH WKH WJU.DQVIRUDRQGWLRQ.QZKHQ FRPSDUHG ZLWK WK.H SWDQQDSKXQFWLRQ ZLWK LWV SDUHQW IXQFWLRQ RQ WKH VD

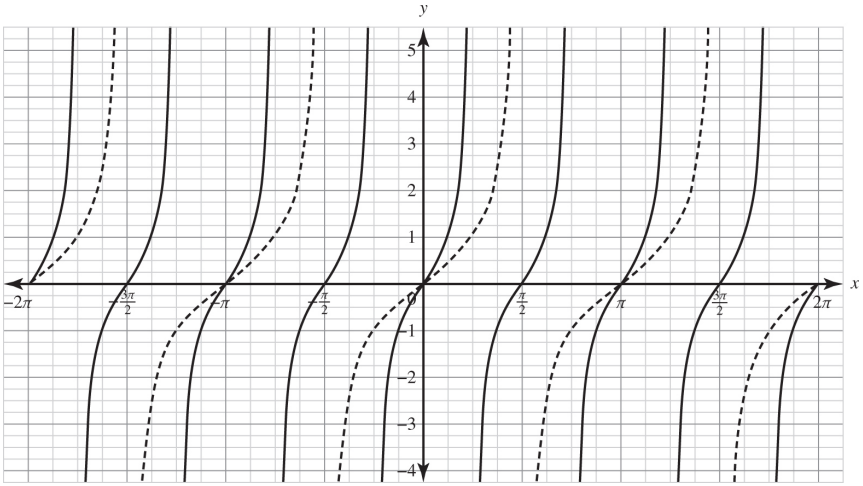
6ROXWLRQ

7KH JUDS.K RIW.DQ D YHUWLFDO GLO7KHWLJRU RJ... LV VROLG DQG. WVKHGJ\RSG HQ



)LJXUH

7KH JUDSK RI D QLQH WDQJHQW IXQFWLRQ LV VKRZQ LQ WKH GLDJUDP DERYH
D IDFWRU RI $QVZHU JUDSK LV VKRZQ LQ WKH GLDJUDP
)LJXUH



)LJXUH

*UDSK RI DW DEQ F G

7KH JUDSK DERYH [Q F G LV D YHUWLFDO GLOD WDQJHQW IXQFWLRQ D IDFWRU RI LV D YGHU

XQLWV DQG LVFDXGSLKWDVH VKLIW RI

› ([DPSOH

*LYHQ WKH $y = 2\tan 3\left(x + \frac{\pi}{4}\right) + 1$ RQV WDWH LWV SDUHQ
WUDQVIRUPDWLRQV RI WKH JLYHQ HTXDV

✓ 6ROXWLRQ

7KH SDUHQW .XQFWLRQ WDQRHQ JLVY HQ HTXDWL\RQ
DWE[Q F G ZKHDU E $c = \frac{\pi}{4}$ DQG G

7KH JU $y = 2\tan 3\left(x + \frac{\pi}{4}\right) + 1$ LV D YHUWLFDO VWDEQDI
IDFWRU RI K$\frac{\pi}{3}$ DX XQDLWSWUKRGV RI YHUWLFDC
SKDVH VKLI$\frac{\pi}{4}$WXQLWRWKH OHIW

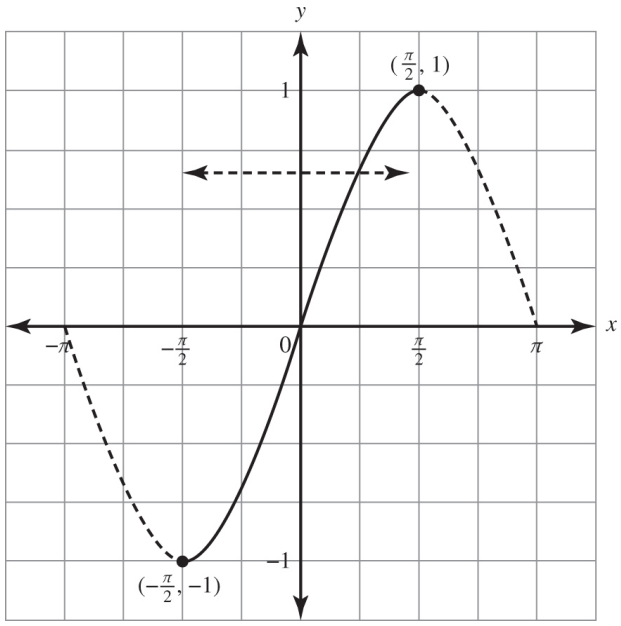
,QYHUVH 7UJRQRPHWULF)XQFWLRQV

6LQFH WKH JUDSKV RI WKH VLQH FRVLQ
KRUL]RQWDO OLQH WHVW QRQH RI WKHV
WULJRQRPHWULF IXQFWLRQV FDQ EH IRU
RULJLQDO WULJRQRPHWULF IXQFWLRQV
IXQFWLRQV SDVV WKH KRUL]RQWDO OLQH

)RU LQYHUVH WULJRQRPHWULF IXQFWL
VZLWFKHG IURP WKHLU FRUUHVSRQGLQJ
YDOXH RI DQ LQYHUVH WULJRQRPHWULF
PHDVXUH DQG WKH LQSXW LV D YDOXH L

,QYHUVH 6LQH)XQFWLRQ

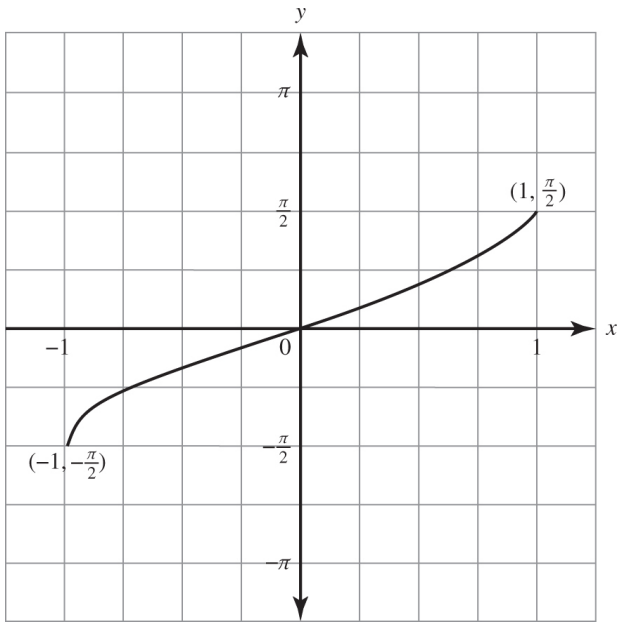
6LQFH VLQH LV D SHULRGLF IXQFWLRQ L
UHVWULFWHG$$ WKH LQWHUYDO SRDVV WK
OLQH WHVW$ LQ WKH LQWHUYDO RQ LWV IXOO UDQJH
DV VKRZQ LQ



)LJXUH *UDSK RI $\sin^{-1} x$

2YHU WKH UHVWULFWHG GRPDLQ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ WKH
UHSUHVHQWV WKH [YDOXH WKDW \RX FDQ DOV
WKH DQJOH ZKRVH VLQH LV

7R JUDSK $\sin^{-1} x$ ZLWK WKH LQSXW DQG R
\ VLQ[ ] DQG SORW)LJXUH RQ LQ



)LJXUH *UDSK RI DUFWDQ

7KH JUDSK RI DUFWDQ EH XVHG WR HYDOXDWH VSHFLILF YDOXHV DQG VR FDQ WKH XQL

([DPSOH

*LYHQ DUFWDQ WKH IROORZLQJ

$f\left(\frac{1}{2}\right)$

$f\left(-\frac{\sqrt{3}}{2}\right)$

6RROXWLRQ

7KH JUDSK RI DUFWDQ EH XVHG WR HYDOXDWH LQSXW YDOXHV LQ DQG GRZQVWKHKVHR OXW7LKRHC XQLW FLUFOH FDQ DOVR EH XVHG Z\KHUH FRRUGLQDWH RI WKH SRLQW RI WKH XQL DVVRFLDWHG ZLWK WKDW JLYHQ SRLQW

)RU $f\left(\frac{1}{2}\right)$ WKH JUDSK RI DQ EH XVHGLW RZH WHOO WKH RXWSXW YDOXH VSHFLILFD EHWZHHQ $\frac{\pi}{4}$ ZKI $x=\frac{1}{2}$ G 8VLQJ WKH XQLW FLUFOH WKH DQJOH $\frac{1}{2}$ ZKRVH $\frac{\pi}{6}$ GHE WKH FRRUGLQDWH RI WKH SRLQW RQ WKH XQLW FLUFOH LV $\left(\frac{1}{2}\right.$ 7KHUHVSUDHQVHGLQJ WDQJHQW RI $\frac{1}{2}$ LV $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ GRPDLQ RI

)RU $f\left(-\frac{\sqrt{3}}{2}\right)$ XWLOL]LQJ WKH XQLW FLUFOH DQG ZKRVH $-\frac{\sqrt{3}}{2}$ 4 $\frac{5\pi}{3}$ V RU LWV FRW $-\frac{\pi}{3}$ 7KHQ DOV $-\frac{\pi}{3}$ J WKHFRRUGLQDWH WKH $-\frac{\sqrt{3}}{2}$ 7UHSUD $-\frac{\pi}{3}$ VDQGQWR ZLWKLQ WKH UHV $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ LRW HVG GHRPDLQ RI

,QYHUVH &RVLQH DQG WDQJHQW)XQFWLRQ

6LQFH FRVLQH DQG WDQJHQW DUH DOVR RQO\ LI WKHLU GRPDLQV DUH UHVWULFWHG WKUHH LQYHUVH IXQFWLRQV

7DEOH , QVHJURVQHRPHWULF)XQFW

, QYHUVH)XQ5HVWULFWHG 5RRWDLQLQFWHG
\ V9L[Q > 9 @ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
\ F9R[V > 9 @ > • @
\ W9DQ 9 ; ; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

([DPSOH

, II [V[LJQ[F9R[V DQG[G W9DQH YDOXDWH WK

$g\left(\frac{\sqrt{3}}{2}\right)$

J 9

K 9

$f\left(g\left(\frac{\sqrt{3}}{2}\right)\right)$

6ROXWLRQ

$g\left(\frac{\sqrt{3}}{2}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ since $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

J 9 9FRV • VLQFH FRV • 9

$h(-1) = \tan^{-1}(-1) = -\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right) = \tan\left(\frac{7\pi}{4}\right) = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1.$

$f\left(g\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$

([DPSOH

, II [VLQ [DUFWDQ LV WKH" YDOXH RI

6ROXWLRQ

$f(1) = \sin(\arctan(1)) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

› ([DPSOH

:KDW LV W $tan(arccos(−√3/2))$?H RI

✓ 6ROXWLRQ

)LUVW $arccos(−√3/2)$ XD7WHH UHVWULFWHG UDQJHRH IR
WKH DQJ $cosθ = −√3/2$ <RXVHH WKH XQLW FLUFOH W
ZKR[VHR RUG $√3/2$ Q7DKWL M LUWH IHUH $π/6$ Q F6HL QDFQHJ WHK HL VF
QHJDWLYH WKH DQJOH7RXVWIRUHLQ LQ4HXI
$θ = π − π/6 = 5π/6$ 1RZ HY $tan(5π/6)$ XD8WHLQJ WKH XQLW FLU
$$\tan\left(\frac{5\pi}{6}\right) = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

7HFKQRORJ\

$ JUDSKLQJ FDOFXODWRU FDQ DOVR EH X
IXQFWLRQV DOWKRXJK WKH DQVZHUV ZL
H[DFW YDOXHV %H VXUH WKH FDOFXODW

› ([DPSOH

$SSUR[$sin⁻¹(3/7)$ WWR WKH QHDUHVW WKRXVDQ(

✓ 6ROXWLRQ

6LQFH WKHUH LV QR SRLQW $3/7$ QHDUHVW WKH
ZD\ WR ILQG WKH $3/7$ QLJVO HW ZKRVHHSDWE EHXC
VHH WKDW WKH FDOFXODWRU LV VHW WR
(17(57KH FDOFXODWRU VKRXOGLYGLDSSOUDR\|
FDQ EH URXQGHG WR UDGLDQV

7UJRQRP HWULF (TXDWLRQV DQG ,QHTXDOLWLHV

,QYHUVH WULJRQRPHWULF IXQFWLRQV DU
LQHTXDOLWLHV LQYROYLQJ WULJRQRPHW
PLQG WKDW VLQFH WKH LQYHUVH WULJR
VROXWLRQV WR HTXDWLRQV DQG LQHTXD

7R VROYH D WULJRQRPHWULF HTXDWLR
IXQFWLRQ LQ WKH DQJOH NHHSLQJ LQ
IXQFWLRQ LV SRVLWLYH RU QHJDWLYH L
WULJRQRPHWULF IXQFWLRQV DUH SHULR
LQILQLWH VROXWLRQV

› ([DPSOH

6ROYH[VLQ [IRZLKHU[HB B

✓ 6ROXWLRQ

)LUVW VROYH IRU WKH WULJRQRPHWULF

$$2\sin(x) + 1 = 0$$

$$\sin x = -\frac{1}{2}$$

1H[W VROYH [IRXLQJHLQYHUVHWULJRQR
UHIHUHQFH DQJOH

$$\sin x = -\frac{1}{2}$$

$$x = \sin^{-1}\left(-\frac{1}{2}\right)$$

7KH UHIHUH $\frac{\pi}{6}$)VHLQDFQHJDWLYHWK LV FRKRHUH $\frac{1}{2}$)QDO
6LQFH VLQH LV QHJDWLYHLQ4XDGUHDQ
VROXWLRQV 7DEVOHKRZQ LQ

WULJRQRPHWULF HTXDWLRQ ,W LV EHV
ZKLFK LV WKH DQJOH LQ 4XDGUDQW , I
TXDGUDQW WR ILQG WKH DQJOH RI WKH
WKH YDOXH DQG WKHQ XVH WKH LQYHUVH
FDOFXODWRUV HYDOXDWH WKH LQYHUV
XVLQJ D SRVLWLYH LQSXW YDOXH HYH

7DEOH

4XDGUDQW 4XDGUDQW ,9	
$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$	$x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

%RWK VROXWLRQV FKHFN RXW ZKHQ ORR
[B • +RZHYHU WKH UHIHUHQFH DQJOH
RQO\ WKH DQJOHV LQ 4XDGUDQW , WR IL

([DPSOH

6ROYH [WDQ [Q ZKHUH • B [I U R WKH QHDUH
GHJUHH

6ROXWLRQ

)LUVW VROYH IRU WKH WUEJRQRPHWULF
TXDGUDWLF HTXDWLRQ LQ VWDQGDUG IR

$$\tan^2 x - \tan x - 2 = 0$$

$$(\tan x + 1)(\tan x - 2) = 0$$

$$\tan x + 1 = 0 \text{ or } \tan x - 2 = 0$$

$$\tan x = -1 \text{ or } \tan x = 2$$

1H[W VROYH HDFK RI WKHVH WULJRQRPH
DQJOHV DQG ZKRVH

7DEOH

[W9DQ	[W9DQ
8VLQJ WKH XQLW FLUFOH DQJOH WKDW LV ZKRVH $\frac{y}{x}=1$WLR RI	8VLQJ WKH UHIHUHQFH DQJOH L
6LQFH WDQJHQW LV 4XDGUDQWV, DQG SRVVLEOH VROXWLRQV	6LQFH WDQJHQW LV 4XDGUDQWV , DQG , SRVVLEOH VROXWLRQ
4XDGUDQW , $x=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$	4XDGUDQW , [UHIHUHQFH DQJO
4XDGUDQW ,9 $x=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$	4XDGUDQW , , , [• ?

7KH IRXU SRVVLEOH VROXWLRQV $x=\frac{3\pi}{4},\frac{7\pi}{4},1.1,4.2$ KLQ UDGLDQV

› ([DPSOH

)LQG WKH H[DFW $\sqrt{2}\cos\left(x-\frac{3\pi}{4}\right)+1=0$.LRQV RI

✓ 6ROXWLRQ

)LUVW VROYH IRU WKH $\cos\left(x-\frac{3\pi}{4}\right)$ JRQRPHWULF

$$\cos\left(x-\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

1H[W VROYH $\text{angle}\left(x-\frac{3\pi}{4}\right)$ XVLQJ LQYHUVH WULJRQ UHIHUHQFH DQJOH

$$\left(x-\frac{3\pi}{4}\right)=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

8VLQJ WKH XQLW FLUFOH VKRZ WKDW WKH HTXDWLRQ
[FRRUG$QDWH LV

6LQFH FRVLQH LV QHJDWLYH LQ 4XDGUDQWV ,, DQG ,,, WKH LQWHUYDO > •@

7DEOH

4XDGUDQW ,,	4XDGUDQW ,,,
$QJOH IRU	$QJOH IRU
$VROXWLRQ IRU	$VROXWLRQ IRU

6LQFH WULJRQRPHWULF IXQFWLRQV DUH SHULRGLF WKHUH DUH LQILQLWHO\ PDQ\ VROXWLRQV WR WULJRQRPHWULF HTXDWLRQV LQ JHQHUDO 7KH H[DPSOH GLG QRW LQFOXGH D VSHFLILF LQWHUYDO VR WKH VROXWLRQV FDQ EH H[SUHVVHG XVLQJ DQ DUELWUDU\ LQWHJHU N •

7KH VROXWLRQ $QG WKH • N • ZKHUH N LV DQ LQW

([DPSOH

)LQG WKH H[DFW VROXWLRQV RI $QG WKH VROXWLRQV

6ROXWLRQ

,VRODWH WKH WULJRQRPHWULF IXQFWLRQ

5HZULWH WKH WULJRQRPHWULF H[SUHVVV

6R OYH IRU[WKH DQJOH

$$x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

8VLQJ WKH XQLW FLUFOH HVWDEOLVKHV WKDW WKH DWUHQ UDW $\frac{y}{x} = \frac{1}{\sqrt{3}}$ RI

6LQFH WDQJHQW LV SRVLWLYH LQ 4XDGU VROXWLRQV LQ WKH LQWHUYDO > •@

7DEOH

4XDGUDQW	,4XDGUDQW
[UHIHUHQFH angle $= \frac{\pi}{6}$)	F H $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

6LQFH WULJRQRPHWULF IXQFWLRQV DUH VROXWLRQV WR WULJRQRPHWULF HTXDWLRQV H[DPSOH GLG QRW LQFOXGH D VSHFLILF VROXWLRQV FDQ EH LQFOXGHG DQG YUHL N•

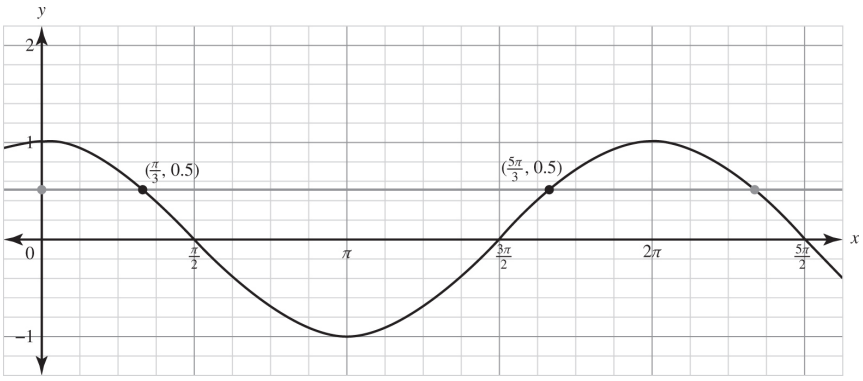
7KH VROX $x = \frac{\pi}{6} + k2\pi$) YD $x = \frac{7\pi}{6} + k2\pi$ ZKHNULHV DQ LQWI

› ([DPSOH

6RO Y[H JILRYUHQ WKH $\cos\left(2x + \frac{\pi}{4}\right) < \frac{1}{2}$ K DZ KHWH[B B•

✓ 6RO XWLRQ

,QHTXDOLWHLUHVQDWUHUBPIHTXDWLRQV LQ WLQ D VROXWLRQ VHW)LUVW ILQ($\frac{1}{2}$ Z&KRHQVH WKH JUDSKRI RR $y = \frac{1}{2}$ DV VKRZQ U



)LJXUH

7KH JUDSKV L $x = \frac{\pi}{3}$ HDU $\frac{5\pi}{3}$ G7KW ZKHS RI FRVLQ
JUDS $y = \frac{1}{2}$ RE EHWZHHQ WKRVH YDOXHV

7KHUHIRUH WKH IROORZLQJ LQHTXDOL

$$\frac{\pi}{3} < 2x + \frac{\pi}{4} < \frac{5\pi}{3}$$

6RO Y[H IRU

$$\frac{\pi}{3} - \frac{\pi}{4} < 2x < \frac{5\pi}{3} - \frac{\pi}{4}$$

$$\frac{\pi}{12} < 2x < \frac{17\pi}{12}$$

$$\frac{\pi}{12} \cdot \frac{1}{2} < x < \frac{17\pi}{12} \cdot \frac{1}{2}$$

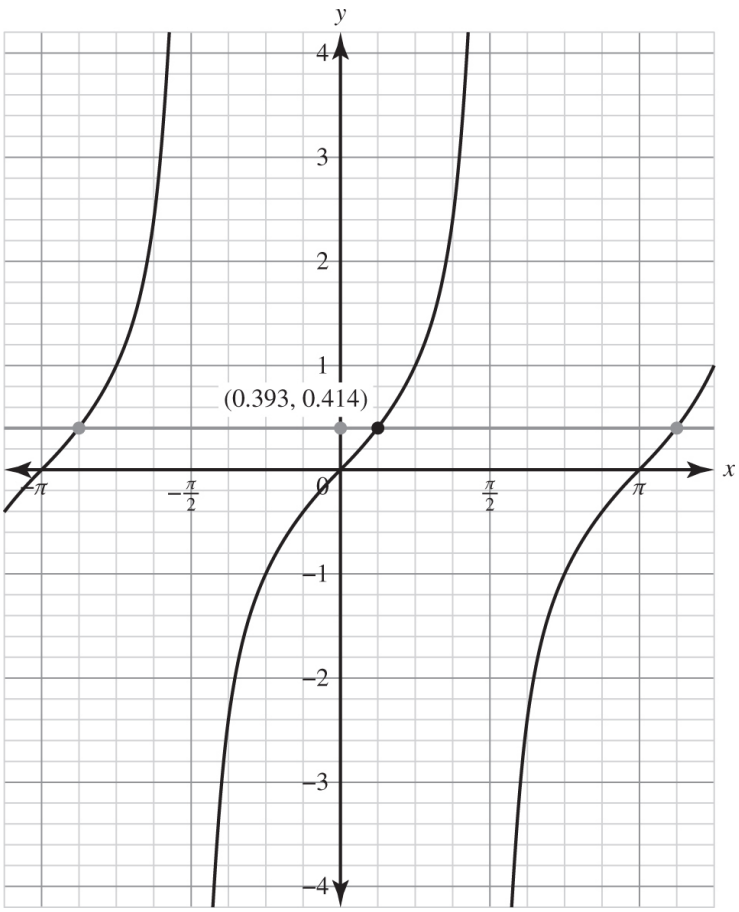
$$\frac{\pi}{24} < x < \frac{17\pi}{24}$$

([DPSOH

6RO Y[H JILRYHQ WKH L[QHTXDOLW\ WDQ

6RO XWLRQ

)LUVW ILQG ZKHUHEWDVHQ ZKHUH WKH FXUYH LV EHORZ WKH [RDL[RVQWKRXQ ILQG



)LJXUH

8VLQJ WKH [JUDSK LW DSSHDUV WKDW WKH DQJOH LV MXVW WRR VPDOO WKHUHIRUH WKH IROORZLQJ LQHTXDOLW\

$$-\frac{\pi}{2} < x < 0.393$$

6LQFH QR GRPDLQ ZDV VSHFLILHG WKHUH UHSHDWHG IRU HYHU\ F\FOH RI •

7KHUHIRUH WKH DQJOHV WKDW VDWLVI\ $-\frac{\pi}{2} + k\pi < x < 0.393 + k\pi$ ZKHUH N LV DQ LQWHJHU

7KH 6HFDQW &RVHFDQW DQG &RWDQJHQW)XQFWLRQV

7KHUH DUH D WRWDO RI V7LK[HWWUHLPRQQRPQJ
UHFLSURFDOV RI WKH EDVLF WULJRQRPH

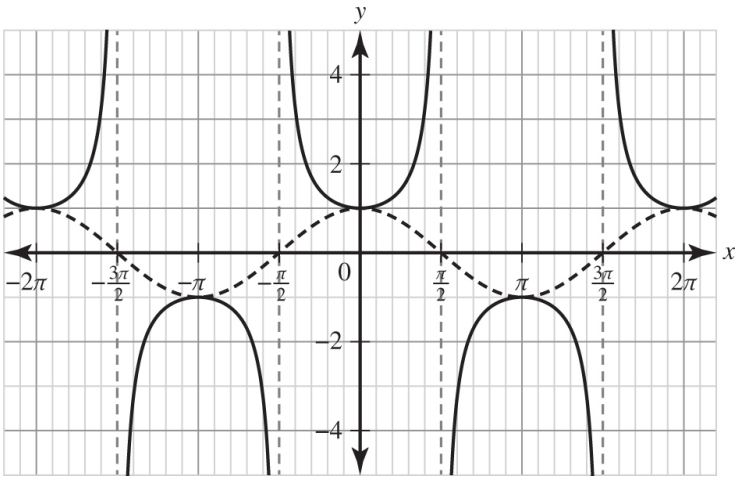
7KH 6HFDQW)XQFWLRQ

7KH VHFDQW...IXQYLHWLRQ KH UHFLSURFDO R
ZKHUH..@RR TGHWHUPLQH..WKH..HRSKDEDH
ZKLFK VKRZV YDOXHV RI ERWK WKH FRVL

7DEOH

	9 •	$-\frac{3\pi}{2}$	9 •	$-\frac{\pi}{2}$		$\frac{\pi}{2}$	•	$\frac{3\pi}{2}$	•
FR									
VH	$\frac{1}{1}=1$	$\frac{1}{0}$ XQGHILQ	$\frac{1}{-1}=-1$	$\frac{1}{0}$ XQGHILQ	$\frac{1}{1}=1$	$\frac{1}{0}$ XQGHILQ	$\frac{1}{-1}=-1$	$\frac{1}{0}$ XQGHILQ	$\frac{1}{1}=1$

$W [WKDOXHV ZKHUH WKH VHFDQW IXQFW
DV\PSWRWHVU DISKV VRHFQG LWV IU..HFE3RH
GLVSOD\HG RQ WKH)LJXUHVHW RI D[HV L



)LJXUH *UDSKV RMHFDQG FRV; • [DD •

7KH IROORZLQJ DUH NH\ FKDUDFWHULVW
IURP WKH JUDSK

- 7KH SHULRG RI WKH VHFDQW IXQFWLF
  •
- :KHQHYHU [HTXDOV [VHF
- 7KH JUDSK VKRZV DV \PSWRWHV DW ZKL
  [
- $ ORFDO PD[LPXP RFFXUV DW D F
  VHFDQG D ORFDO PLQLPXP RFFXUV ZLW
  PD[LPXP RU[HF
- 7KH UDQJH RI WKH VHFDQW IXQFWLRQ

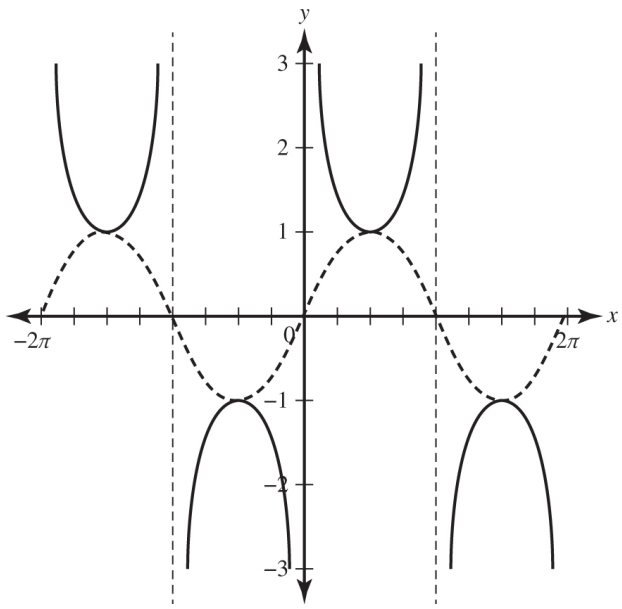
7KH &RVHFDQW)XQFWLRQ

7KH FRVHFDQW IXQFWLRQ LV UHFLSURFDO
ZKHUH LW LV GHILQHG WKH VLQH
ZKLFK VKRZV YDOXHV RI ERWK WKH VLQH

7DEOH

	9 •	$-\frac{3\pi}{2}$	9•	$-\frac{\pi}{2}$		$\frac{\pi}{2}$	•	$\frac{3\pi}{2}$	•
VL									
FV	$\frac{1}{0}$ XQGHILQHG	$\frac{1}{1}=1$	$\frac{1}{0}$ XQGHILQHG	$\frac{1}{-1}=-1$	$\frac{1}{0}$ XQGHILQHG	$\frac{1}{1}=1$	$\frac{1}{0}$ XQGHILQHG	$\frac{1}{-1}=-1$	$\frac{1}{0}$ XQGHILQHG

$W [ZKHUH WKH FRVHFDQW IXQ
DV \PSWRWHV 7KH JUDSK FRQG LWV IUHF
GLVSOD\HG RQ WKH)LJXUH VHW RI D[HV L



)LJXUH *UDSKV RI VHFDQ\G VLQ; • [DD •

7KH IROORZLQJ DUH NH\ FKDUDFWHULVWLFV VHHQ IURP WKH JUDSK

- 7KH SHULRG RI WKH FRVHFDQW IXQFWL •
- :KHQHYHU [LV HTXDO WR LQWHJHU [
- 7KH JUDSK FRQWDLQV DV\PSWRWHV ZKHQ [
- $ ORFDO PD[LPXP RFFXUV ZKHQ D FRV[F DQG D ORFDO PLQLPXP RFFXUV ZKHQ PD[LPXP RI VHF[
- 7KH UDQJH RI WKH FRVHFDQW ;IXQFWL

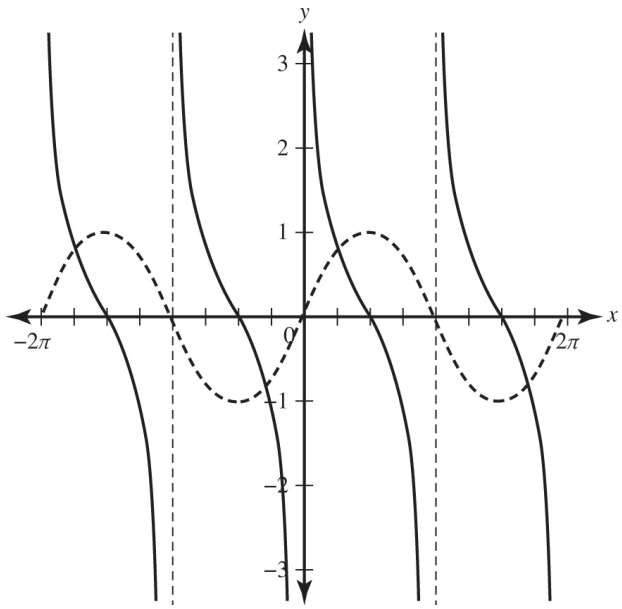
7KH &RWDQJHQW)XQFWLRQ

7KH FRWDQJHQW IXQFWLRQ LV UHFLSURFDO ZKHUH LV DQ (TXLOLQH RI V GHWHUPLQ RI ... F..RWRQ VDEOH ZKLFK VKRZV YDOXHV DQG FRWDQJHQW IXQFWLRQV

7DEOH

	9 •	$-\frac{3\pi}{2}$	9 •	$-\frac{\pi}{2}$		$\frac{\pi}{2}$	•	$\frac{3\pi}{2}$	•
FR									
VL									
FR	$\frac{1}{0}$ XQGH	$\frac{0}{1}=0$ IL	$\frac{-1}{0}$ QXHQGG H	$\frac{0}{-1}=0$ IL	$\frac{1}{0}$ QXHQGG H	$\frac{0}{1}=0$ IL	$\frac{-1}{0}$ QXHQGG H	$\frac{0}{-1}=0$ IL	$\frac{1}{0}$ QXHQGG HI

$W WKH OXHV ZKHUH WKH FRWDQJHQW IXDV\PSWRWHV RFFXU DQG GLVSOD VHW RI JUDSKV LQ



)LJXUH *UDSKV RI WDQ\ VLQ; • [DD •

7KH IROORZLQJ DUH NH\ FKDUDFWHULVWLFV VHHQ IURP WKH JUDSK

- 7KH SHULRG RI WKH FRWDQJHQW IXQI LV •
- 7KH JUDSKF[RWDV YHUWLFDO DV\PS$WRRU ZKLFK[VLQV/ LQcot $x = \frac{\cos x}{\sin x}$R W
- 7KH JUDSK RI WKH FRWDQJHQW IXQFV FRQVHFXWLYH YHUWLFDO DV\PSWRWH
- 7KH FRWDQJHQW IXQFWLRQ OLNH WK

7R HYDOXDWH WKH UHFLSURFDO WULJR UHFLSURFDO IXQFWLRQV VLQH FRVLQH

› ([DPSOH

)LQG WKH H[DFW YDOXH RI HDFK RI WKH

$\sec\left(\frac{\pi}{6}\right)$

$\csc\left(\frac{\pi}{2}\right)$

$\cot\left(\frac{7\pi}{4}\right)$

F R9V VHF •

✓ 6RO XWLRQ

$$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\csc\left(\frac{\pi}{2}\right) = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = \frac{1}{1} = 1$$

$$\cot\left(\frac{7\pi}{4}\right) = \frac{1}{\tan\left(\frac{7\pi}{4}\right)} = \frac{1}{-1} = -1$$

$$\cos^{-1}(\sec\pi) = \cos^{-1}\left(\frac{1}{\cos(\pi)}\right) = \cos^{-1}\left(\frac{1}{-1}\right) = \cos^{-1}(-1) = \pi + k2\pi$$

(TXLYDOHQW5 HSUHVHQWDWLRQV RI 7UJRQRPHWULF)XQFWLRQV

,PSRUWDQW WULJRQRPHWULF IRUPXODV ZRUNLQJ ZLWK WULJRQRPHWULF HTXDWLRQV

LGHQWLW\ WKDW KDV $\tan\theta = \frac{\sin\theta}{\cos\theta}$ D7GK\L WHHTX DGW\
H[DPSOH RI D WULJRQRPHWULF LGHQWLW
VXEVWLWXWLRQ WKH H[SUHVVLRQV D

3\WKDJRUHDQ ,GHQWLWLHV

7KH 3\WKDJRUHDQ WKHRUHP FDQ EH DSS
WKH XQLW FLUFOH..DWL.QFRRUGLQDWHJHRPH
LGHQWLW\

/HW[\ UHSUHVHQW DQ\ SRLQW RQ WKH
&RQVWUXFWLQJ D ULJKW WULDQJOH
IURP S3RWLQW[WR WKH [LV JLYHV E\ WKH 3\WKDJR
6LQ3FHV RQ WKH[XFRWDQGLQH6XEVWLW..XWH
DQG..VLQSURGXFHV WKH 3\WKD.JRUHDQ LG

'LYLGLQJ HDFK WHUP RI WKH 3\W.KDJR
OHDGV WR WZR DGGLWLRQDO 3\WKDJRUH
(DFK RI WKH WKUHH 3\WKDJRUHDQ WULJ
PRUH WKDQ RQH ZD\

7DEOH 3\WKDUJRUHQDQP7WULF ,GH

7KUHH 3\WKDJRUHDQ 7ULJRQRPHWULF ,GHQWLWLHV	)R
■ VLQ. FRV. ■ WDQ VHF. ■ FRW FVF..	■ VLQ. ; FRV RU FRV.. ; VLQ ■ WDQ VHF.. ; RU VHF.. ; WDQ ■ FRW. FVF.. ; RU FVF.. ; FRW

3\WKDJRUHDQ7RPHWULF ,GHQW

V L Q. F R V.

([DPSOH

5HZULWH WKH H[SUHVVLRQ LQ WHUPV RI

6RXWLRQ

$ FRPPRQ VWUDWHJ\ WR UHZULWH WULJR
LQWR WKH EDVLF WULJR.QDPHWRVLF IXQF\

$$\cos\theta(\sec\theta - \cos\theta) = \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right)$$

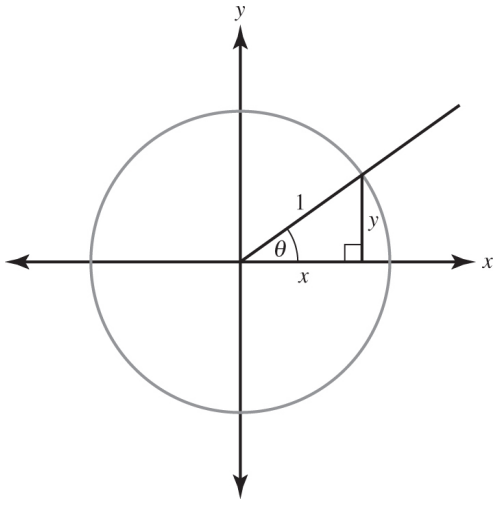
'LVWULEXWH

$$= 1 - \cos^2\theta$$

8VH WKH 3\WKDJRUHDQ LGHQWLW\

$$= \sin^2\theta$$

7KH 3\WKDJRUHDQ LGHQWLWLHV FDQ D
HVWDEOLVK UHODWLRQVKLSV DPRQJ WUL
WULJRQRPHWULF IXQFWLRQV)LJXUH LZKHUH
K\SRWHQXVH LV DQG OHJV DUH



)LJXUH

,QVWHDG RI UHIHUULQJ WR WKH 3RWKDJRU
VROYH IRU WKH[OHJ LQ WHUPV RI

$$x^2 + y^2 = 1^2$$

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1 - x^2}$$

6LQFH WKLV WULDQJO $y = +\sqrt{1 - x^2}$ GUDZQ LQ 4XD
7KH IROORZLQJ WZR WULJRQRPHWULF I

$$\sin\theta = \frac{\sqrt{1 - x^2}}{1}$$

DQG

$$\cos\theta = \frac{x}{1}$$

8VLQJ LQYHUVH WULJRQRPHWULF ..IXQFWI

$$\theta = \arcsin\sqrt{1 - x^2}$$

DQG

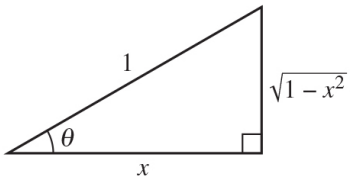
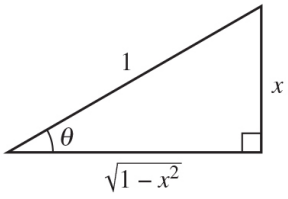
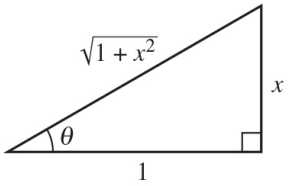
$$\theta = \arccos x$$

6LQFH ERWK DUH UHIHU.UWLQH WR RWHTXDO
HTXDO

$$\arccos x = \arcsin\sqrt{1 - x^2}$$

,Q WKH ULJKW WULDQJOH WKH VLGHQVE
UHDUUDQJHG DIHOURZLQJ WKRUHP ZLWK VLGH [WR I
VKRZVHGHQW LQYHUVH WULJRQRPHWULF
7DEOH 7KH WDEOH DVVXPHWLWKDW DQG G
TXDGUDQW WKH WULDQJOH LV GUDZQ LQ
IXQFWLRQV

7DEOH

Diagram	Relationships
	■ $\arccos x = \arcsin\sqrt{1-x^2}$ ■ $\arccos x = \arctan\frac{\sqrt{1-x^2}}{x}$
	■ $\arcsin x = \arccos\sqrt{1-x^2}$ ■ $\arcsin x = \arctan\frac{x}{\sqrt{1-x^2}}$
	■ $\arctan x = \arcsin\frac{x}{\sqrt{1+x^2}}$ ■ $\arctan x = \arccos\frac{1}{\sqrt{1+x^2}}$

6LQH DQG &RVLQH 6XP ,GHQWLWLHV

7KHUH DUH LGHQWLWLHV WKDW DOORZ W
GLHUHQFH RI WZR DQJOHV WR EH H[SUHV
WULJRQRPHWULF IXQFWLRQV RI WKH LQG

6XP ,GHQWLWLHV

VLQ ~ • VLQ ~ FRV • FRV ~ VLQ •
FRV ~ • FRV ~ FRV • ; VLQ ~ VLQ •

› ([DPSOH

)LQG WKH H[$\sin\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

✓ 6RO XWLRQ

6LQFH QHL$\frac{\pi}{9}$V K$\frac{2\pi}{9}$U KDQV JNH RZQ FRRUGLQDW
D WULJRQRPHWULF LGHQWLW\ WR ZULWH

DQ DQJOH ZLWK NQRZQ FRRUGLQDWHV R
8VH WKH VXP LGHQWLW\ IRU VLQH

$$\sin\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)=\sin\left(\frac{\pi}{9}+\frac{2\pi}{9}\right)=\sin\left(\frac{3\pi}{9}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

› ([DPSOH

)LQG WKH H[$\cos\left(\frac{7\pi}{12}\right)$ YDOXH RI

✓ 6ROXWLRQ

6LQFH $\frac{7\pi}{12}$ LV DQ DQJOH DQ DQJOH ZLWK NQRZ
XVH D WULJRQRPHWULF LGHQWLW\ WR ZU
LQ DQ DQJOH ZLWK NQRZQ FRRUGLQDWH
6LQ $\frac{\pi}{3}+\frac{\pi}{4}=\frac{4\pi}{12}+\frac{3\pi}{12}=\frac{7\pi}{12}$ XVH WKH IRUPXOD $\alpha=\frac{\pi}{3}$ DQ
$\beta=\frac{\pi}{4}$

$$\begin{aligned}\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)&=\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\&=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\&=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}\\&=\frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

6LQH DQG &RVLQH 'LIIHUHQFH ,GHQW\

7KH VXP LGHQWLWLHV IRU VLQH DQG FRVLQH
LGHQWLWLHV E\ DGGLQJ D QHJDWLYH DQ
7KH GHULYDWLRQ RI WKH GLIIHUHQFH LGHQWLW\ IRU V

$$\sin(\alpha-\beta)=\sin(\alpha+-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$

6LQFH WKH JUDSK RI VLQH LV V\PPHWULF
RGG IXQFWLRQ PHDQLQJ WKDW VLQ 9 ~
V\PPHWULF ZLWK [DQG [VR FRV LV
WKDW FRV 9 ~ FRV ~
6XEVWLWXWH WKLV LQWR WKH SUHYLRX\

$$\sin(\alpha-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$
$$\sin(\alpha-\beta)=\sin\alpha\cos(\beta)+\cos\alpha\cdot-\sin(\beta)$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

7KH GHULYDWLRQ IRU FRVLQH RI WKH GLIIHUHQFH RI WZR DQJOHV LV

$$\cos(\alpha-\beta)=\cos(\alpha+-\beta)=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)$$

6XEVWLWXWH WKH HYHQ DQG RGG IXQFWL

$$\cos(\alpha-\beta)=\cos\alpha\cos(\beta)-\sin\alpha\cdot-\sin(\beta)$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

'LIIHUHQFH ,GHQWLWLHV

VLQ ; • VLQ ~ FRV • ; FRV ~ VLQ •
FRV ; • FRV ~ FRV • VLQ ~ VLQ •

› ([DPSOH

, $\alpha=\frac{\pi}{6}$ D $\beta=\arccos\frac{3}{5}$ ZKDW LV WKH H[DFW YDOXH

✓ 6RO XWLRQ

6LQ $\sec(\alpha-\beta)=\frac{1}{\cos(\alpha-\beta)}$ ILUVW ILQG WKH H[DFW YD
$\alpha=\frac{\pi}{6}$ WKDW $\alpha=\frac{1}{2}$DQG $\cos\alpha=\frac{\sqrt{3}}{2}$Q

, $\beta=\arccos\frac{3}{5}$ WKHQ LQ 4XDGUD $\cos\beta=\frac{3}{5}=\frac{x}{r}$ ZKHUH
[\ DQG $\sin\beta=\frac{y}{r}=\frac{4}{5}$

8VH WKH GLIIHUHQFH LGHQWLW\ IRU FRVLQH

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3}{5}\right)+\left(\frac{1}{2}\right)\left(\frac{4}{5}\right)$$
$$=\frac{3\sqrt{3}+4}{10}$$

6LQ $\cos(\alpha-\beta)=\frac{3\sqrt{3}+4}{10}$, $\sec(\alpha-\beta)=\frac{10}{3\sqrt{3}+4}$

'RXEOH $QJOH ,GHQWLWLHV

7KH VXP LGHQWLWLHV IRU VLQH DQG FRV
DQJOH LGHQWLWLHV IRU VLQH DQG FRVL
7KH GHULYDWLRQ RI WKH GRXEOH DQJ(

$$\sin(2\alpha) = \sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$$

&RPELQH OLNH WHUPV

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

7KH GHULYDWLRQ RI WKH GRXEOH DQJ(

$$\cos(2\alpha) = \cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

8VH WKH 3\WKDJRUHDQ LGHQWLWLHV WR

$$\cos(2\alpha) = 1 - \sin^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha$$

8VH WKH 3\WKDJRUHDQ LGHQWLWLHV WR

$$\cos(2\alpha) = \cos^2\alpha - (1 - \cos^2\alpha) = \cos^2\alpha - 1 + \cos^2\alpha = 2\cos^2\alpha - 1$$

'RXEOH $QJOH ,GHQWLWLHV

VLQ ~ VLQ ~ FRV ~
FRV ~ FRV VLQ
FRV ~ ; ~ VLQ
FRV ~ FRV

› ([DPSOH

*LY $\sin x = \frac{3}{4}$ ILQG WKH H[DFW YDOXH RI FRV

✓ 6ROXWLRQ

6LQFH WKH [YDOXHV RI VLQ[DQG FRV[DUH WKH VDPH
LQ WKH ILUVW DQG IRXUWK TXDGUDQWV RQO\ VLQ[LV SRVLWLYH

$$\cos(2x) = 1 - 2\left(\frac{3}{4}\right)^2 = -\frac{1}{8}$$

3URYLQJ 7ULJRQRPHWULF ,GHQWLWLHV

3URSHUWLHV RI WULJRQRPHWULF IXQFWL
RWKHU DOJHEUDLF SURSHUWLHV FDQ EH
LGHQWLWLHV 3URYLQJ WKDW D WULJRQ
VKRZLQJ WKDW WKH WZR VLGHV RI WKH
DOLNH

6WUDWHJLHV UWLR3QURRPYHW7UL ,GHQ

$ IHZ VWUDWHJLHV FDQ EH DSSOLHG ZK
LGHQWLWLHV

- 6WDUW ZLWK WKH PRUH FRPSOLFDW
  H[SUHVV LW XVLQJ RQO\ VLQHV DQG
-)DFWRU
- &RPELQH DQG VLPSOLI\ IUDFWLRQDO
- 0DNH D VXEVWLWXWLRQ XVLQJ D NQ
  VXFK DV D TXRWLHQW UHFLSURFDO

,I QRQH RI WKH DERYH VWUDWHJLHV ZR
RWKHU VLGH RI WKH HTXDWLRQ WR VLQ

([DPSOH

3URYH WKDW WKH IROORZLQJ HTXD.WRLLRQ
WKH H[SUHVVLRQV DUH GHILQHG

$$(\cot\theta + \csc\theta)(1 - \cos\theta) = \sin\theta$$

✓ 6RO XWLRQ

6WDUWLQJ ZLWK WKH OHIW VLGH RI WKH UHZULWH WKH WHUPV XVLQJ RQO\ VLQH

$$(\cot\theta + \csc\theta)(1 - \cos\theta) = \left(\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right)(1 - \cos\theta)$$

&RPELQH DQG VLPSOLI\ WKH IUDFWLRQ

$$= \left(\frac{\cos\theta + 1}{\sin\theta}\right)(1 - \cos\theta)$$

0XOWLSO\ WKH IUDFWLRQV

$$= \frac{1 - \cos^2\theta}{\sin\theta}$$

6XEVWLWXWH XVLQJ D 3\WKDJRUHDQ LGH

$$= \frac{\sin^2\theta}{\sin\theta}$$

6LPSOLI\

$$= \sin\theta$$

› ([DPSOH

3URYH WKDW WKH IROORZLQJ HTXDWLRQ WKH H[SUHVVLRQV DUH GHILQHG

$$\sec\theta - \cos\theta = \sin\theta\tan\theta$$

✓ 6RO XWLRQ

5HZULWH WKH WHUPV RQ ERWK VLGHV RI

$$\frac{1}{\cos\theta} - \cos\theta = \sin\theta \cdot \frac{\sin\theta}{\cos\theta}$$

6LPSOLI\ WKH ULJKW VLGH RI WKH HTXDWL

$$\frac{1}{\cos\theta} - \cos\theta = \frac{\sin^2\theta}{\cos\theta}$$

7KH QHZ JRDO LV WR VKRZ WKH OHIW VL
V L $\frac{\sin^2\theta}{\cos\theta}$
&RPELQH DQG VLPSOLI\ WKH IUDFWLRQD

$$\frac{1}{\cos\theta} - \cos\theta = \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta} = \frac{1 - \cos^2\theta}{\cos\theta}$$

6XEVWLWXWH XVLQJ D 3\WKDJRUHDQ LGH

$$= \frac{\sin^2\theta}{\cos\theta}$$

%RWK VLGHV RI WKH $\frac{\sin^2\theta}{\cos\theta}$TXDWLRQ DUH HTXDO

6RONLQJ WULJRQRPHWULF (TXDWLRQV

3UHYLRXVO\ \RX VKRZHG WKDW VROYLQJ WU
WZR VWDJHV)LUVW WKH WULJRQRPHWULF
LV VROYHG DQG WKHQ WKH VROXWLRQV DUH XVHG WR ILQG WKH YDOXHV RI WKH XQNQRZQ YDULDEOH

:KHQ SUHVHQWHG ZLWK D WULJRQRPHWULF IXQFWLRQ LQ D VXEVWLWXWLRQ RU RWKHUZLVH LW LV XVHIXO WR UHFRJQLVH WKDW LW LV D TXDGUDWLF LQ WHUPV RI RQH RI WKH WULJRQRPHWULF UDWLRV

([DPSOH

6ROYH IRU WKH QHDUHVW WHQWK RI D GHJUHH ZKHUH [B B•

6ROXWLRQ

6LQFH WKH HTXDWLRQ FRQWDLQV RQO\ WULJRQRPHWULF LGHQWLWLHV LW LV QHFHVVDU\ WR H[SUHVV DOO WHUPV LQ WHUPV RI RQH UDWLR 8VLQJ D 3\WKDJRUHDQ LGHQWLW\ FDQ EH XVHIXO IRU

$$3\cos^2 x + 5\sin x = 4$$
$$3(1 - \sin^2 x) + 5\sin x = 4$$
$$3 - 3\sin^2 x + 5\sin x = 4$$

7KLV LV QRZ D TXDGUDWLF HTXDWLRQ W

$$3\sin^2 x - 5\sin x + 1 = 0$$

)LUVW VROYH IRU WKH W[UXLVJLRQQJRWWKHWUL[FD

$$\sin x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)}$$

$$\sin x = \frac{5 \pm \sqrt{13}}{6}$$

7DEOH VKRZV KRZ WR VROYH IRU DQJOH

7DEOH

$x = \sin^{-1}\left(\frac{5+\sqrt{13}}{6}\right)$ $x \approx \sin^{-1}(1.4343)$ 7KHUH LV QR SRVVLEOH KHUH VLQFH WKH PD[LV	$x = \sin^{-1}\left(\frac{5-\sqrt{13}}{6}\right)$ $x \approx \sin^{-1}(0.2324)$ UHIHUHQFH DQJOH 6LQH LV SRVLWLYH I , , 4XDGUDQW , [UHIHUHQFH DQJ UDGLDQV 4XDGUDQW , , [A • ; U

7KH HTXDWLRQ KDV WZR VROXWLRQV

([DPSOH

)LQG DOO [YLDQOWKH RQBWH•UWDKDW B/DWLVI\ [9 F[RV

6ROXWLRQ

6LQFH WKH HTXDWLRQ LQHFRVQWDSLQW PZDRQG LGHQWLW\ LV QHHGHG WR UHZUL WH WKH 8VH D GRXEOH DQJOH LGHQWLW\ WR UH IRUP

$$\cos(2x) - \cos x = 0$$

$$(2\cos^2 x - 1) - \cos x = 0$$

$$2\cos^2 x - \cos x - 1 = 0$$

)LUVW VROYH IRU WKH W[UEL\JRDQFRWPRHUWLQLF

$$(2\cos x + 1)(\cos x - 1) = 0$$

$$\cos x = -\frac{1}{2},\ \cos x = 1$$

7DEOH VKRZV KRZ WR V[ROYH IRU DQJOH

7DEOH

$x = \cos^{-1}\left(-\frac{1}{2}\right)$ 7KH UHIHU=$\frac{\pi}{3}$QFH DQJOH &RVLQH LV QHJDWLY7KLQ 4XSHUQWW ,W DQG ,,, 4XDGUDQW ,, $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ 4XDGUDQW ,,, $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$	[FRV DQJOHV D[Q G •

7KH VROXWLRQ $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi.$ DUH

0 XOWLSOH &KRLFH 4 XHVWLRQV

_ :KLFK RI WKH IROORZLQJ FKRLFHV LV I
~ VHF ~ 9 FRV ~ "

$ FRV~

% FRV ~ 9 VLQ ~

& VLQ~

' FRW ~ 9

_ ,I WKH WHUPLQDO VLGH RI DQ DFXWH DQJOH $ LQ VWDQGDUG SRVLWLRQ LV SDUDOOHO WR WKH OLQH $\cos A = \frac{5}{13}$ ZKDW LV WKH YDOXH RI VLQ

$ $\frac{144}{65}$

% $\frac{60}{65}$

& $\frac{144}{169}$

' $\frac{60}{156}$

_ ,I[LV D SRVLWLYH D[FXWH DQJOH DQG VLQ I H[SUHVVLRQ LQ WHUPV RI

$ $\frac{1-a}{a}$

% $\sqrt{1-a^2}$

& $\frac{\sqrt{1-a^2}}{a}$

' $\frac{1}{1-a}$

_ , $\tan x = \frac{\sqrt{3}}{3}$ D $\cos x = -\frac{\sqrt{3}}{2}$ ILQG WKH «P[HDVXUH RI

$ $\frac{2\pi}{3}$

% $\frac{5\pi}{6}$

& $\frac{7\pi}{6}$

' $\frac{5\pi}{3}$

7KH JUDSK RI ZKLFK HTXDWLRQ DRQG KDV SHDU

$ \ V[Q

% $y = 2\sin\frac{1}{2}x$

& $y = \frac{1}{2}\sin 2x$

' $y = \frac{1}{2}\sin\frac{1}{2}x$

:KLFK RI WKH IROORZLQJ H[SUHVVLRQ

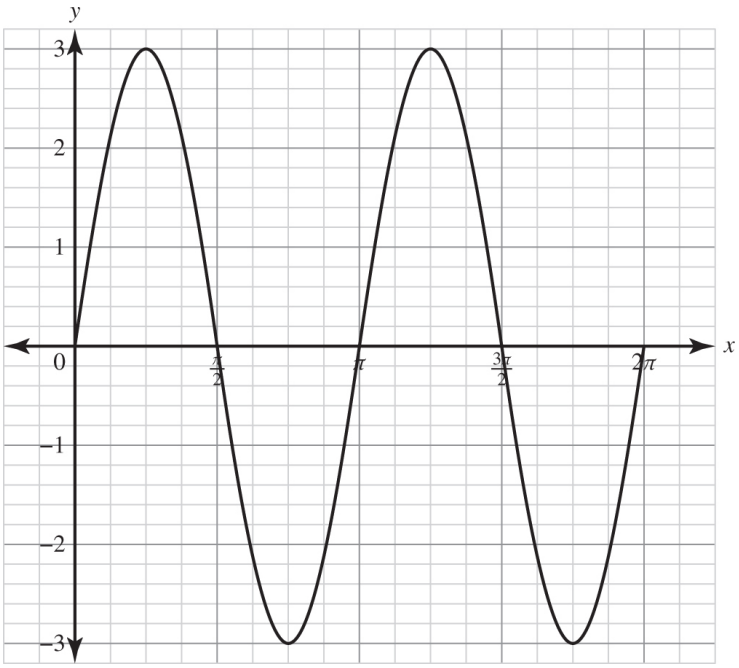
$ FRV.

% $\cos\left(\theta + \frac{\pi}{2}\right)$

& $\cos\left(\theta - \frac{\pi}{2}\right)$

' FRV.9

:KLFK RI WKH IROORZLQJ HTXDWLRQV EHORZ"



$ \ V[Q

% \ V[Q

& \ V[LQ

' \ V [Q

$ W Z K L F K [Y G R H X V W W K $f(\theta)=\tan\left(\theta-\frac{\pi}{4}\right)$ K R D Y H D Y H D V \ P S W R W H "

$ $-\frac{\pi}{4}$

%

& $\frac{\pi}{4}$

' $\frac{\pi}{2}$

, I [V L Q [F R Z K D W L V $f\left(\frac{1}{2}\right)$ " H Y D O X H R I

$

% $\frac{1}{2}$

& $\frac{\sqrt{2}}{2}$

' $\frac{\sqrt{3}}{2}$

, $\cos x=\frac{12}{13}$, $\sin y=\frac{4}{5}$ D Q G D U H D F X W H D Q J O H V V L [Q 9 \ "

$ $\frac{72}{65}$

% $\frac{56}{65}$

& $-\frac{16}{65}$

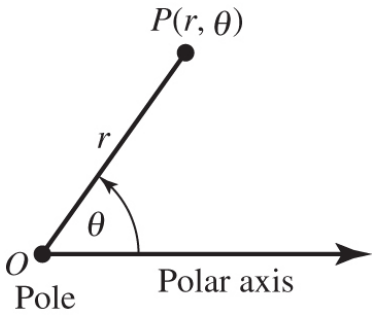
' $-\frac{33}{65}$

$ QVZ HU ([SODQDWRQV

& 5 H Z U L W H W K H H [S U H V V L R Q L Q W H U P

$$\cos\alpha(\sec\alpha-\cos\alpha)=\cos\alpha\left(\frac{1}{\cos\alpha}-\cos\alpha\right)$$
$$=1-\cos^2\alpha=\sin^2\alpha$$

$ % \ W K H 3 \ W K D J R U V H Q D H R V G H Q Z K L W F K J L $\sin^2 A+\left(\frac{5}{13}\right)^2=1.$ 6 $\sin^2 A=1-\frac{25}{169}=\frac{144}{169}.$ 6 L Q F H $ D V Q H J O H L Q D W



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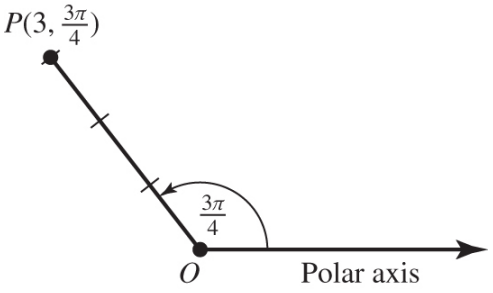
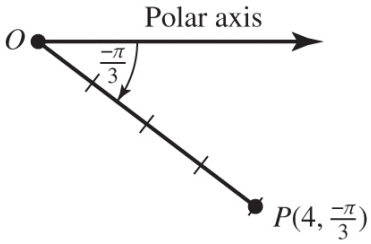
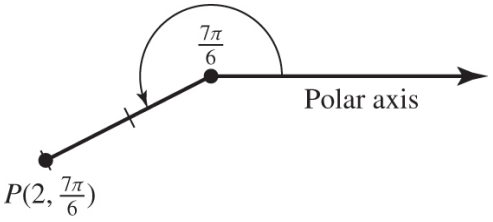


$P\left(3, \frac{3\pi}{4}\right)$

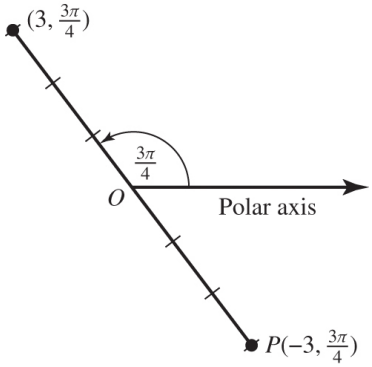


$P\left(4, \frac{-\pi}{3}\right)$

■



$P\left(2, \frac{7\pi}{6}\right)$



$P\left(-3, \frac{3\pi}{4}\right)$

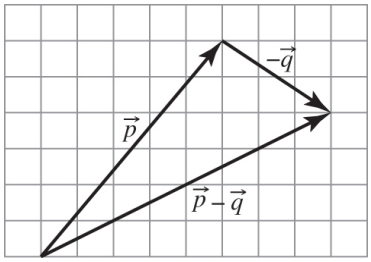
›

$C\left(5, \frac{7\pi}{6}\right)$
$M\left(5, \frac{19\pi}{6}\right)$
$P\left(-5, \frac{\pi}{6}\right)$

✓

)LJXUH

*UDSK WKH VXPE\RIFRQQHFWLQWRWKHWK
7KLV LV VJKRXOG



)LJXUH

6FDODU 3URGXFWV

7KHUH DUH PDQ\ SUDFWLFDO DSSOLFDWL
ZD\ WR ILQG WKH SURGXFW RI WZR YHFW
YHFWRUV RU W $\vec{a} = \langle a_1, a_2 \rangle$ RD $\vec{b} = \langle b_1, b_2 \rangle$ GXW $\vec{a} \cdot \vec{b}$ HQ ZKLFK
UHDGGLRWF DQ EH IRXQG XVLQJ WKH IROOI

$$\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2$$

6LQFH WKH GRW SURGXFWUHDWXOWZKLQL
VFDODU 7SKLRVGLXFWHQW IURP VFDODU PXOW
PXOWLSOLHG E\ D YHFWRU \LHOGLQJ D Q

3URSHUWLHV RI WKH 'RW 3UR

7KH GRW SURGXFW KDV WKH IROORZLQ

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2$$

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$$

$$(\vec{a}+\vec{b})\cdot(\vec{c}+\vec{d}) = \vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{d} + \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{d}$$

([DPSOH

, $\vec{p} = \langle 2, 3\rangle$ $\vec{q} = \langle 5, -1\rangle$ D $\vec{r} = \langle 4, -2\rangle$ ILQG HDFK RI WKH IRO

$\vec{p}\cdot\vec{q}$

$2\vec{r}\cdot\vec{p}$

$\vec{p}\cdot(\vec{q}+\vec{r})$

6ROXWLRQ

$$\vec{p}\cdot\vec{q} = 2\cdot 5 + 3\cdot -1 = 10 - 3 = 7$$

$$2\vec{r}\cdot\vec{p} = \langle 8, -4\rangle\cdot\langle 2, 3\rangle = 8\cdot 2 + -4\cdot 3 = 16 - 12 = 4$$

$$\vec{p}\cdot(\vec{q}+\vec{r}) = \vec{p}\cdot\vec{q} + \vec{p}\cdot\vec{r} = (2\cdot 5 + 3\cdot -1) + (2\cdot 4 + 3\cdot -2) = 7 + 2 = 9$$

*HRPHWULF 3URSHUWLHV RI WKH 6FL

7KH GRW SURGXFW RI WZR YHFWRUV KDV

, I...LV WKH DQJOH $\vec{a}$ EHWZHHQ $\vec{b}$ WRUV $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ WRUV

)RU QRQ]HUR YHFWRUV $\vec{a}$ DQG $\vec{b}$

- $\vec{a}\cdot\vec{b} < 0$ LI DQG RQO\ LI WKH DQJOH EHWZHHQ $\vec{a}$ DQG $\vec{b}$ LV REWXVH
- $\vec{a}\cdot\vec{b} > 0$ LI DQG RQO\ LI WKH DQJOH EHWZHHQ $\vec{a}$ DQG $\vec{b}$ LV DFXWH
- $\vec{a}\cdot\vec{b} = 0$ LI DQG RQO\ LI $\vec{a}$ DQG $\vec{b}$ DUH SHUSHQGLFXODU
- $\vec{a}\cdot\vec{b} = \pm|\vec{a}||\vec{b}|\cos\theta$ LI DQG RQO\ LI $\vec{a}$ DQG $\vec{b}$ DUH QRQ]HUR S

([DPSOH

, $\vec{a} = \langle -1, 3 \rangle$ D $\vec{b} = \langle 0, 2 \rangle$ I L Q G W K H I R O O R Z L Q J

D $\vec{a} \cdot \vec{b}$

E 7 K H D Q J O H $\vec{a}$ E $\vec{b}$ W Z H H Q

✓ 6 R O X W L R Q

D $\vec{a} \cdot \vec{b} = \langle -1, 3 \rangle \cdot \langle 0, 2 \rangle = (-1)(0) + (3)(2) = 0 + 6 = 6$

E 6 L Q F H W K H G R W S U R G X F W L V J U H D W H
Y H F W R U V L V D F X W H

$$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}||\vec{b}| \cos\theta \\ 6 &= \sqrt{(-1)^2 + (3)^2}\,\sqrt{(0)^2 + (2)^2} \cos\theta \\ 6 &= \sqrt{10}\,\sqrt{4} \cos\theta \\ 6 &= 2\sqrt{10} \cos\theta \\ \cos\theta &= \frac{6}{2\sqrt{10}} \end{aligned}$$

7 K H Y D O X H V R I F R V L Q H D U H S R V L W L Y
µ p p @ + 2 Z H D Y O U H D G \ N Q R Z W K H
W K H U H I R U H O L P L W H G W R W K H L Q W H U

$$\theta = \arccos\frac{6}{2\sqrt{10}} \approx 18.43^\circ$$

› ([D P S O H

) L Q G V X F K $\vec{p} = \langle x, -3 \rangle$ D W W $\vec{q} = \langle 7, -4 \rangle$ D U H S H U S H Q G L F X O I

✓ 6 R O X W L R Q

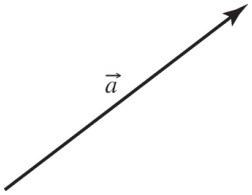
7 Z R Y H F W R U V D U H S H U S H Q G L F X O D U L I W
S U R G X F W E H W Z H H Q W K H W Z R Y H F W R U V

$$\begin{aligned} \vec{p} \cdot \vec{q} &= \langle x, -3 \rangle \cdot \langle 7, -4 \rangle = (x)(7) + (-3)(-4) = 7x + 12 \\ 7x + 12 &= 0 \\ x &= \frac{-12}{7} \end{aligned}$$

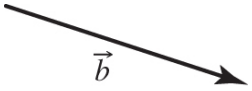
$ Q J O H % H W Z H H Q W R U V

,I WZR QRQ]HUR YHFWRUV HPDQDWH IURP
EHWZHHQHWFKDQP XVH WKH ODZ RI FRVLQHV
DQJOH

,PDJLQH \RX DUH J $\vec{a} = \langle a_1, a_2 \rangle$ DQG $\vec{b} = \langle b_1, b_2 \rangle$ FDWWRUVKP
)LJXUHVDQG

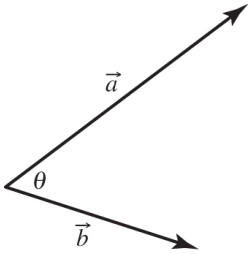


)LJXUH



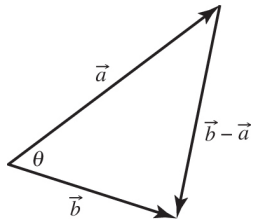
)LJXUH

:H FDQ WUDQVODWH WKH WZR YHFWRUV
WR IRUP DQ JVKRZQ LQ



)LJXUH

&RQQHFWLQJ WKH WDLOV RI $\vec{a} + \vec{b} = \vec{b} - \vec{a}$ RU
LV VKRZQ LQ



)LJXUH

7KH OHQJWK RI HDFK YHFWRU FDQ EH IRXQG XVLQJ WKH ODZ RI FRVLQHV

$$|\vec{b}-\vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

&DOFXODWLQJ DQG VLPSOLI\LQJ WKH GHWDLOV \LHOGV WKH IRUPXOD WR ILQG WKH DQJOH EHWZHHQ WZR YHFWRUV

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2}{|\vec{a}||\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

› ([DPSOH

)LQG WKH DQJOH EHWZHHQ $\vec{u} = \langle 3, -5\rangle$ DQG $\vec{v} = \langle -4, -2\rangle$

✓ 6ROXWLRQ

7R ILQG WKH DQJOH LW LV HDVLHU WR FDOFXODWH WKH YHFWRUV VHSDUDWHO\ DQG WKHQ V

$$\vec{u}\cdot\vec{v} = (3)(-4) + (-5)(-2) = -12 + 10 = -2$$

$$|\vec{u}| = \sqrt{(3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

$$|\vec{v}| = \sqrt{(-4)^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20}$$

6XEVWLWXWH LQWR WKH IRUPXOD

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{-2}{\sqrt{34}\sqrt{20}}$$

$$\theta = \arccos\frac{-2}{\sqrt{34}\sqrt{20}} \approx 94.40^\circ$$

8QLWYHFWRUV

$ XQLW YHFWRU LV DQ\ YHFWRU WKDW KI
YHFWRUV DUH VKRZQ WR WKH ULJKW DV

([DPSOHV RI8QLW9HFWRUV

LV D XQLW YHFWRU VLQFH LWV PD

$$\sqrt{(1)^2 + (0)^2} = \sqrt{1 + 0} = \sqrt{1} = 1$$

$\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ LV D XQLW YHFWRU VLQFH LWV PD

$$\sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

› ([DPSOH

)LQG D XQLW YHFWRU LQ WKH IROORZLQJ XQLW Y
W

$\left\langle t, -\frac{2}{3} \right\rangle$

✓ 6RO XWLRQ

)RU W WR EH D XQLW YHFWRU LWV PDJQLWXGH PXVW

$$\text{Magnitude} = \sqrt{(1)^2 + (t)^2} = \sqrt{1 + t^2}$$

6HW WKH PDJQLWXGH HTXDO WR DQG

$$\begin{aligned} \sqrt{1 + t^2} &= 1 \\ \left(\sqrt{1 + t^2}\right)^2 &= (1)^2 \\ 1 + t^2 &= 1 \\ t^2 &= 0 \\ t &= 0 \end{aligned}$$

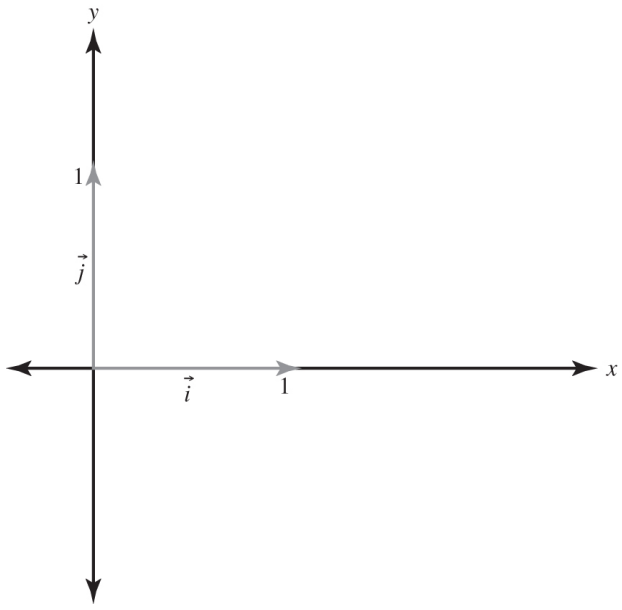
)R $\left\langle t, -\frac{2}{3} \right\rangle$ WR EH D XQLW YHFWRU LWV PDJQLWXGH PXVW

$$\text{Magnitude} = \sqrt{(t)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{t^2 + \frac{4}{9}}$$

6HW WKH PDJQLWXGH HTXDO WR DQG

$$\sqrt{t^2 + \frac{4}{9}} = 1$$
$$\left(\sqrt{t^2 + \frac{4}{9}}\right)^2 = (1)^2$$
$$t^2 + \frac{4}{9} = 1$$
$$t^2 = \frac{5}{9}$$
$$t = \pm\sqrt{\frac{5}{9}} = \pm\frac{\sqrt{5}}{3}$$

6SHFLDO XQLW YHFWRUV $\vec{i} = \langle 1, 0 \rangle$ UDQG $\vec{j} = \langle 0, 1 \rangle$ 7RKHWWRKHU H LQ WKH GLUHFWLRQV RI WKH SRVLWLYH [DQG \ D[HV



)LJXUH

7KLV DOVR DOORZV XV WR ZULWH YHFWRU YHFWRUV $\vec{a} = \langle a_1, a_2 \rangle$ P WKLV LV ZULWWHQ LQ FRP HTXLYDOHQW $\vec{a} = a_1\vec{i} + a_2\vec{j}$ WZR KLFK LV ZULWWHQ LQ XQ

([DPSOH

([SUHVV WKH IROORZLQJ YHFWRUV LQ FR

$$\vec{p} = -3\vec{i} + 5\vec{j}$$
$$\vec{q} = \frac{1}{2}(\vec{i} - 6\vec{j})$$

✓ 6 R O X W L R Q

7 K H F R P S R Q H Q W $\vec{p} = \langle -3, 5 \rangle$ I R & U D P O H F W X O D W H W K H O

$$\vec{p} = |\vec{p}| = \sqrt{(-3)^2 + (5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

7 K H F R P S R Q H Q W $\vec{q} = \left\langle \frac{1}{2}, -3 \right\rangle$ R U & D O F X O D W H W K H O

$$\vec{q} = |\vec{q}| = \sqrt{\left(\frac{1}{2}\right)^2 + (-3)^2} = \sqrt{\frac{1}{4} + 9} = \sqrt{\frac{37}{4}}$$

6 R P H W L P H V L W L V X V H I X O W R I L Q G D X D J L Y H Q Y H F W R U

8 Q L W Y F W R U D Q G D * L Y H Q ' L U H F W

) R U D Q \ Q R Q] H U R Y H F W R U $\vec{u}$ W K D W K D V W K H G L U H F W L R Q D V $\vec{p}$ E X W K D V D O H Q W W R

$$\vec{u} = \frac{\vec{p}}{|\vec{p}|}$$

› ([D P S O H

) L Q G D X Q L W Y H F W R U $\vec{p} = 2\vec{i} - 7\vec{j}$ K H V D P H G L U H

✓ 6 R O X W L R Q

) L U V W I L Q G W H H P D J Q L W X G H R I

$$|\vec{p}| = \sqrt{(2)^2 + (-7)^2} = \sqrt{4 + 49} = \sqrt{53}$$

0 X O W L S O \ W K H $\frac{1}{\sqrt{53}}$ ' L V W U L E X W H G L U H F W L R Q D O X Q L W Y H F W R U

$$\vec{u} = \frac{1}{\sqrt{53}}\left(2\vec{i} - 7\vec{j}\right) = \frac{2}{\sqrt{53}}\vec{i} - \frac{7}{\sqrt{53}}\vec{j}$$

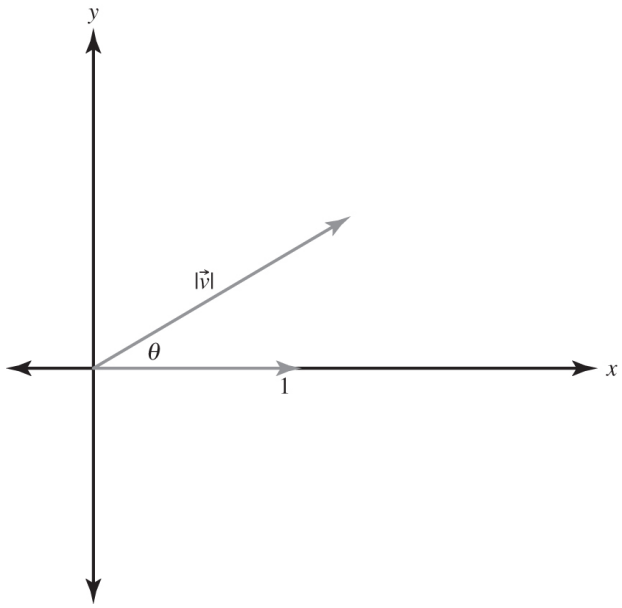
7R F K H F N L W K D W Q L W K R E F W K R U W W K H V P H D J Q L

$$|\vec{u}| = \sqrt{\left(\frac{2}{\sqrt{53}}\right)^2 + \left(-\frac{7}{\sqrt{53}}\right)^2} = \sqrt{1} = 1$$

) L Q G L Q H F W R U I U R P , W V ' L U H F W L R Q

, W L V P R U H F R P P R Q I R U Y H F W R U V W R E H P D J Q L W X G H U D W K H U W K D Q W K H L E P R S F R S H U D W L R Q V Z H U H F D O F X O D W H G L W L Q J L W R O H D U Q K R Z W R I L Q G W K H F R P S R Q H Q P D J Q L W X G H

6 X S S R V H Z H D U H J L Q W K H D Q J O H W Z H H G W K H



) L J X U H

, Q J H Q H U D O W K H V D P H G L U H F W L R Q J K D W W U L D Q J O I R U P $\vec{v} = |\vec{v}|\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right)$

([DPSOH

$ EDOO LV WKURZQ ZLWK DQ LQLWLDO VS
PDNHV DQ DQJOH RI [pD[ZLWK([WKHVSRWLV
LQ WHUPV RI

6RO XWLRQ

)LUVW ILQG WH6FLDQJFQHLWXGHVRHG ZDV J
KRXU DQG LV WKH VF7KHDQXJOHRIEYHWC
DQG WKH[SRLVWLVY..Hp pV R

)LQG WKH YHORFLW\ YHFWRU

$$\vec{v} = 20\left(\cos 45°\,\vec{i} + \sin 45°\,\vec{j}\right) = 20\left(\frac{\sqrt{2}}{2}\,\vec{i} + \frac{\sqrt{2}}{2}\,\vec{j}\right) = 10\sqrt{2}\,\vec{i} + 10\sqrt{2}\,\vec{j}$$

9HFWRU9DOXHG)XQFWLRQV

3DUDP 0RWLRQDO9HG)XQFWLRQV

$Q LPSRUWDQW DSSOLFDWLRQ RI YHFWR
UHSUHVHQWHG E\ SDUDPHWULF HTXDWLF

3RVLWLRQ DQG 0DJQLWXGH

7KH SRVLWLRQ RI D SDUWLFOH PRYLQJ L
IXQFWLRQ[QW\W FDQ EH H[SUHVVHGXEHG DX
$\vec{p}(t) = \langle x(t), y(t)\rangle$ F$\vec{p}(t) = x(t)\vec{i} + y(t)\vec{j}$ $W WLWHK Y WJLQQRL
SDUWLFOH LV WR WKH ULJKW RW LQHGWFR
SDUWLFOH LV DERYH RU EHORZ WKH RUL
7KH PDJQLWXGH RI WKH VLRVWHRGL YWE
SDUWLFOH IURP WKH RULJLQ

([DPSOH

$Q REMHFW LV PRYLQJ ZKHUH WKH SRVL
SDUDPHWULF HTXDWLRQV DQG ZKHUI

DUH PHDVXUHG LQ PHWHUV DQG WLPH LV
YDOXHG IXQFWLRQ WKDW UHSUHVHQWV \
GLVWDQFH RI WKH SDUWLFOH IURP WKH

✓ 6ROXWLRQ

: ULWH WKH SDUDPHWULF HTXDWLRQ WR ILQG WKH SRVLWLRQ

$$\vec{p}(t) = \langle t^2 - t, t^2 + t \rangle = (t^2 - t)\vec{i} + (t^2 + t)\vec{j}$$

$W WKH SRVLWLRQ RI WKH SDUWLFOH

$$\vec{p}(2) = \langle (2)^2 - (2), (2)^2 + (2) \rangle = \langle 2, 6 \rangle$$

7KH PDJQLWXGH RI WKH YHFWRU LV WKH

$$|\vec{p}(t)| = \sqrt{(2)^2 + (6)^2} = \sqrt{4 + 36} = \sqrt{40} \text{ meters}$$

9HORFLW\

7KH YHORFLW\ RI D SDUWLFOH FDQ EH LQWHS
WKH YHFWRU IXQFWLRQ $\vec{v}(t) = \langle x(t), y(t) \rangle$ RU $\vec{v}(t) = x(t)\vec{i} + y(t)\vec{j}$ $W WKH
VLJQ WR LQGLFDWHV LI WKH SDUWLFOH LV
\ WLQGLFDWHV LI WKH SDUWLFOH LV LQ
YHORFLW\ YHFWRUV DW WKH VSHHG RI WKH S

› ([DPSOH

$ SDUWLFOH PRYLQJ LQ D SODQH $\vec{v}(t) = \langle 2t, 2t + 1 \rangle$ D
ZKHUH GLVWDQFHV DUH PHDVXUHG LQ $W
KRXUV GHVFULEH WKH GLUHFWLRQ WK
WKH SDUWLFOH

✓ 6ROXWLRQ

$W

$$\vec{v}(4) = \langle 2(4), 2(4) + 1 \rangle = \langle 8, 9 \rangle$$

6LQ[FH ! WKH SDUWLFOH LV\PRYLQJ SDUWLFOH LV PRYLQJ XS

7KH VSHHG RI WKH SDUWLFOH LV IRXQG YHORFLW\WYHFWRU DW

$$|\vec{v}(4)| = \sqrt{(8)^2 + (9)^2} = \sqrt{64 + 81} = \sqrt{145} \text{ miles per hour}$$

› ([DPSOH

$Q REMHFW PRYHV ZL$\vec{s}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$/ LDWQLGRQHYOH YHF$\vec{v}(t) = (-\sin t)\vec{i} + (\cos t)\vec{j}$ ZKHUH GLVWDQFHV DUH P WLPH LV LQ PLQXWHV)LQG WKH GLVWDC VSHHG DW WLPH

✓ 6RO XWLRQ

)LQG WKH GLVWDQFH RI WKH REMHFW IU

$$|\vec{s}(t)| = \sqrt{(\cos t)^2 + (\sin t)^2} = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1 \text{ meter}$$

7KLV PHDQV WKDW WKH GLVWDQFH EHWZHH LV 7KLV PDNHV VHQVH EHFDXVH LI WKH F FUHDWH D XQLW FLUFOH ZKRVH UDGLXV

)LQG WKH VSHHG RI WKH REMHFW

$$|\vec{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1 \text{ meter/minute}$$

7KLV PHDQV WKDW WKH REMHFW PRYHV I

0 XOWLSOH &KRLFH 4XHVWLRQV

_ ,I D SDUWLFOH[\ SRVLWLRQ LV WKH VWDWWLQJ YHF$\vec{v}(t) = \langle t^3 + 2t, t^2 + 1 \rangle$ ZKDW LV WKH VSHHG "R

$ $\sqrt{13}$

% $\sqrt{17}$

&

'

) L Q G W K H N W D X H K $ 9 W D $ 9 D U H S H U S

$

%

&

'

* L Y H Q D S D U W L F O H Z K R L F Q L W H W S H U P L Q X W H Z I K L Q G W K Z H K H Q R W K H V S H H C P H W H U V P L Q X W H

$

%

&

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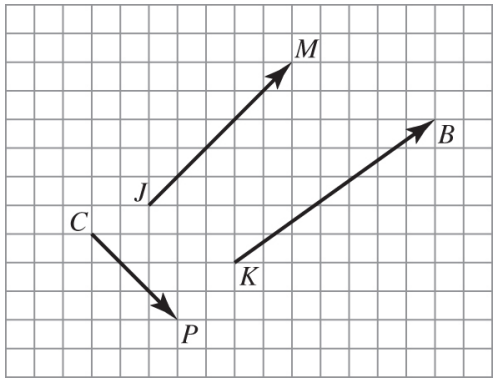
* L Y H Q W K H I L J X R D Z K H L F K R I I R O O R Z L Q J L V H T X L Y D O H Q W W R



$ D G

% D 9G

& 9D G

' 9D 9G

$ SDUWLFOH PRYHV RQ D SODQH LQ VXFK D ZD\ WKDW LWV YHORFLW\ YHFWRU DW DQ\ WLPH W LV JLYHQ E\ $QVZHU WKH IROORZLQJ TXHVWLRQV DERXW WKH PRWLRQ RI WKH SDUWLFOH

L $W ZKDW WLPH GRHV WKH SDUWLFOH UHDFK LWV VORZHVW VSHHG"

$

%

&

'

) LQG WKH [DQG \ FRPSRQHQWV RI WKH DFFHOHUDWLRQ YHFWRU RI D SDUWLFOH ZKRVH YHORFLW\ LV

D &DOFXODWH WKH DQJOH EHWZHHQ WKH YHORFLW\ DQG DFFHOHUDWLRQ YHFWRUV DW W

$ [\

% [\

& [9 \ 9

' [9 \ 9

* LYHQ WKH YHORFLW\ YHFWRUV EHORZ ILQG RQH RI WKH IROORZLQJ

, ,

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$, DQG , , RQO\

% , , DQG , , , RQO\

& , DQG , , , RQO\

& , , , DQG , , ,

* LYHQ WKH YHORFLW\ YHFWRUV EHORZ ILQG RQH RI WKH IROORZLQJ DQJOH EHWZHHQ WKH WZR YHFWRUV

, ,

, ,

, ,

$, RQO\
% ,, RQO\
& ,,, RQO\
' 1RQH RI WKH DERYH

_ *LY $\vec{p}$ DQG WKH DQJOH EHWZHHQ \
IL $\vec{p}\cdot\vec{q}$

$
%
& $\frac{7\sqrt{3}}{2}$
' $\frac{10\sqrt{3}}{2}$

_ :KLFK RI WKH IROORZLQJ LV D YHFWR
9 DQG KDV D OHQJWK RI "
$ 9 9
% 9
&
' 9

$QVZHU([SODQDWLRQV

_ &7KH VSHHG RI D SDUWLFOH LV IRXQG
YHORFLW\ YHFWRU DW WKH JLYHQ WLP
&DOFXODWH WKH YHORFLW\ YHFWRU

$$\vec{v}(2) = \langle (2)^3 + 2(2), (2)^2 + 1 \rangle = \langle 12, 5 \rangle$$

&DOFXODWH WKH VSHHG

$$|\vec{v}(2)| = \sqrt{(12)^2 + (5)^2} = 13$$

_ ' 7ZR YHFWRUV DUH SHUSHQGLFXODU I

$$\vec{c} \cdot \vec{p} = (3)(k) + (10)(-6) = 3k - 60$$

6HW WKH GRW SURGXFW HTXDO WR D

$$3k - 60 = 0$$
$$3k = 60$$
$$k = 20$$

%7KH VSHHG RI D SDUWLFOH LV IRXQG
YHORFLW\ YHFWRU DW WKH JLYHQ WLP

$$\text{Speed} = |\vec{v}(t)|$$
$$\sqrt{10} = \sqrt{(3-t)^2 + (-1+2t)^2}$$
$$\sqrt{10} = \sqrt{5t^2 - 10t + 10}$$

6TXDUH ERWK VLGHV RI WKH HTXDWLR

$$10 = 5t^2 - 10t + 10$$

6HW WKH TXDGUDWLF HTXDO WR DQG

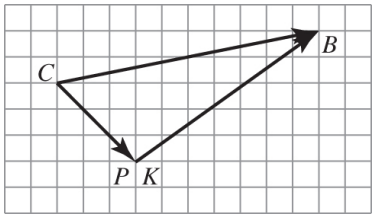
$$0 = 5t^2 - 10t$$
$$0 = 5t(t-2)$$

6HW HDFK IDFWRU HTXDO WR

$$t = 0, t = 2$$

6LQW!H WWK H Q

$ %DVHG RQ WKH ILJXUH $ $\overrightarrow{CB} = \overrightarrow{CP} + \overrightarrow{KB}$ QVLQ FWH
YHFWRUV FDQ EH WUDQ $\overrightarrow{CP}$ FDQ HPH WR $\overrightarrow{KB}$ WK



$$\overrightarrow{CB} = \overrightarrow{CP} + \overrightarrow{KB} = a + d$$

& 7R ILQG WKH VSHHG ILUVW FW OFXOD

$$\vec{v}(1) = \langle 4(1)^3 - 3, 10(5(1) - 2) \rangle = \langle 1, 30 \rangle$$

7KHQ FDOFXODWH WKH VSHHG WKWW P

$$|\vec{v}(1)| = \sqrt{(1)^2 + (30)^2} = \sqrt{901} \approx 30$$

% 7Z R YHFWRUV DU[HDHQTGXRDFOSLRQWWQWLVUDU

$$3x - 2 = y + 5 \text{ and } -y - 3 = 2x - 11$$

6RO YH WKH V\VWHP RI HTXDWLRQV HLV WKH VXEVWLWXWLRQ PHWKRG RU WKH

7R XVH WKH HOLPLQDWLRQ PHWKRG ZU HTXDO VLJQ DQG DGG WKH WZR HTXDW

$$3x - y = 7 \text{ and } 2x + y = 8$$

$$\begin{aligned} 3x - y &= 7 \\ +\,2x + y &= 8 \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

6XEVWLWXWH [WR HLWKHU HTXDWLRQ WR HLWKHU\ HTXDWL

$$\begin{aligned} 2(3) + y &= 8 \\ y &= 2 \end{aligned}$$

& 9HFWRUV DUH SDUDOOHO WR HDFK R RWKHU RQO\ LQ $-\frac{1}{5}$ &KHFN XVLQJ &KRLFH , VEL\Q &KRLFH

$$2\vec{i} - \frac{3}{5}\vec{j} = -\frac{1}{5}\langle -10, 3\rangle$$

$$\vec{c} = -\frac{1}{5}\vec{a}$$

$ 7Z R YHFWRUV IRUP DQ REWXVH DQJO]HU7KH YHFWRUV LQ &KRLFH , IRUP DQ

$$\vec{x} \cdot \vec{y} = (1)(2) + (-2)(3) = 2 - 6 = -4 < 0$$

7KH YHFWRUV LQ &KRLFH ,, IRUP DQ DI

$$\vec{s} \cdot \vec{t} = (1)(2) + (-1)(-3) = 2 + 3 = 5 > 0$$

7KH YHFWRUV LQ &KRLFH ,,, IRUP D UL

$$\vec{a} \cdot \vec{b} = (3)(-4) + (4)(3) = -12 + 12 = 0$$

% %DVHG RQ WKH JLYHQ LQIRUPDWLRQ WZR YHFWRUV LW ZRXOG $\cos\theta = \frac{\vec{p} \cdot \vec{q}}{|\vec{p}||\vec{q}|}$ VLHVW

$$\cos 60° = \frac{\vec{p} \cdot \vec{q}}{(2)(5)}$$

$$\frac{1}{2} = \frac{\vec{p} \cdot \vec{q}}{10}$$

$$\vec{p} \cdot \vec{q} = (10)\left(\frac{1}{2}\right) = 5$$

')LUVW ILQG WKH XQLW YHFWRU LQ FDOFXODWLQJ WKH PDJQLWXGH RI WKH

$$\sqrt{(-5)^2 + (12)^2} = \sqrt{169} = 13$$

7KH $\text{vector} = \frac{1}{13}\langle -5, 12\rangle$

7R KDYH WKH YHFWRU JR LQ WKH RSSR OHQJWK PXOWLSO\ WKH XQLW YHFWRU

$$-26 \cdot \frac{1}{13}\langle -5, 12\rangle = -2\langle -5, 12\rangle = \langle 10, -24\rangle$$

2FHDQRF3RP

12

Matrices

Learning Objectives

In this chapter, you will learn:

- How to organize data in a matrix
- Properties and operations of matrices
- How to use matrices as functions
- How to use matrices to model real-life scenarios

NOTE

These Unit 4 topics are **not** assessed on the AP exam but may be part of your school's course curriculum.

In previous chapters, various math processes were used to solve real-life situations by way of modeling situations and working with data. Being able to organize various equations and data is an important skill. Matrices are a way to do that. A matrix is a set of numbers arranged in rows and columns to form a rectangular array. The numbers are called elements or entries of the matrix. The size, or order, of the matrix is determined by the number of rows by the number of columns, as shown in Table 12.1.

Table 12.1

Matrix	Order
$\begin{bmatrix} 8 \\ 8 \\ 4 \end{bmatrix}$	This matrix has 3 rows and 1 column. We say this is a 3 × 1 matrix; it is read as "3 by 1" matrix. It can also be called a 3 × 1 column matrix or column vector.
$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ -7 & \frac{1}{2} & 15 \end{bmatrix}$	This matrix has 3 rows and 3 columns. It can be called a 3 × 3 matrix. It can also be called a 3 × 3 square matrix since the number of rows is equal to the number of columns.
$\begin{bmatrix} 2 & 10 & 18 & 26 \end{bmatrix}$	This matrix has 1 row and 4 columns. It can be called a 1 × 4 matrix. It can also be called a 1 × 4 row matrix or row vector.

When identifying an element in a matrix, you always list the row and then the column. For example, element *h* is in row 3, column 2.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Matrices are present all around us. A calendar (seen in Figure 12.1) is a form of a matrix. The different dates are elements, the weeks are rows, and the days are columns. Even a recipe (shown in Figure 12.2) can be an example of a matrix with the ingredients as rows and the amounts needed as a column.

July 2001

M	T	W	T	F	S	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

Figure 12.1

Ingredients	Amount
sugar	1 tsp.
flour 1	cup
milk	200 mL
salt	1 pinch

Figure 12.2

12.1 Matrix Operations

This section will discuss the various operations that can be performed with matrices. Many are intuitive and follow the same procedures as operations with equations.

Matrix Equality

Two matrices are equal if they have the same order and the elements in corresponding positions are equal. For example, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then because they are both 2×2 matrices, their corresponding elements are equal: $a = w$, $b = x$, $c = y$, and $d = z$.

Matrix Addition

Two matrices of the same order can be added together by adding their corresponding elements.

Example

If $A = \begin{bmatrix} 6 & -2 & 3 \\ -4 & 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 7 & 9 \\ 10 & 2 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 7 & 3 \\ 5 & -2 \end{bmatrix}$, find the following.

1. $A + B$

2. $A + C$

✓ Solution

1. Since A and B are the same order:

$$A + B = \begin{bmatrix} 6+0 & -2+7 & 3+9 \\ -4+10 & 5+2 & 1+8 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 12 \\ 6 & 7 & 9 \end{bmatrix}$$

2. A and C are of different orders, so those matrices cannot be added together.

Matrix Subtraction

Two matrices of the same order can be subtracted by subtracting their corresponding elements.

› Example

If $A = \begin{bmatrix} 29 & 31 & 19 \\ 13 & 82 & 23 \\ 4 & 7 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 7 & 6 \\ 8 & 20 & 24 \\ 42 & 32 & 21 \end{bmatrix}$, find $A - B$.

✓ Solution

Since A and B are the same order:

$$A - B = \begin{bmatrix} 29-(-1) & 31-7 & 19-6 \\ 13-8 & 82-20 & 23-24 \\ 4-42 & 7-32 & 9-21 \end{bmatrix} = \begin{bmatrix} 30 & 24 & 13 \\ 5 & 62 & -1 \\ -38 & -25 & -12 \end{bmatrix}$$

Matrix Multiplication

To multiply two matrices, the number of columns in the first matrix must equal the number of rows in the second matrix. In other words, the product of an $m \times n$ matrix A with an $n \times p$ matrix B is the $m \times p$ matrix called AB. The new matrix will have the same number of rows as the first matrix and the same number of columns as the second matrix.

For example, if $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and $B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then $AB = \begin{bmatrix} ax+by+cz \\ dx+ey+fz \end{bmatrix}$. Matrix A is order 2×3, matrix B is order 3×1, and matrix AB is order 2×1.

❯ Example

If $A = \begin{bmatrix} 2 & 7 & 5 \\ 3 & 8 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -5 & 0 \\ 2 & 10 \end{bmatrix}$, find AB.

✓ Solution

Since A is a 2×3 matrix and B is a 3×2 matrix, they can be multiplied together since the number of columns in the first matrix is equal to the number of rows in the second matrix. The order of their product matrix will be 2×2.

To multiply, the corresponding elements in the first row of the first matrix are multiplied with the corresponding elements in the first column of the second matrix and then added together. This will be the entry in row 1, column 1. This process continues, multiplying each row from the first matrix with each column in the second matrix as shown below:

$$A \cdot B = \begin{bmatrix} 2 \cdot 1 + 7 \cdot -5 + 5 \cdot 2 & 2 \cdot 3 + 7 \cdot 0 + 5 \cdot 10 \\ 3 \cdot 1 + 8 \cdot -5 + 4 \cdot 2 & 3 \cdot 3 + 8 \cdot 0 + 4 \cdot 10 \end{bmatrix} = \begin{bmatrix} 2 - 35 + 10 & 6 + 0 + 50 \\ 3 - 40 + 8 & 9 + 0 + 40 \end{bmatrix} = \begin{bmatrix} -23 & 56 \\ -29 & 49 \end{bmatrix}$$

Important

Unlike ordinary algebra where multiplication is commutative, matrix multiplication is not commutative since the number of rows and columns might not match if the order of the matrices is reversed.

12.2 The Inverse and Determinant of a Matrix

Identity Matrix

The identity matrix, I, is a square matrix whose elements along the diagonal from the top left to bottom right are 1 and the remaining elements are 0. Examples of two identity matrices of different orders are shown below.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When an identity matrix, *I*, is multiplied by a square matrix, *A*, that multiplication will be commutative and will result in matrix *A*. In other words, $AI = IA = A$. This can be demonstrated below.

If $A = \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix}$ then:

$$AI = \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+5 \\ 7+0 & 0+6 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 2+0 & 5+0 \\ 0+7 & 0+6 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix} = A$$

Inverse Matrix

An $n \times n$ matrix *A* is said to be invertible if there is an $n \times n$ matrix *B* such that their product results in the identity matrix, $AB = I_n = BA$. The matrix *B* is called the inverse of *A* and can be denoted as A^{-1}.

› Example

Verify that matrix $B = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$ is the inverse of matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$.

✓ Solution

To verify that the two matrices are inverses, their product in either order should result in the identity matrix. The product is shown below:

$$AB = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -2+3 & 2-2 \\ -3+3 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2+3 & -1+1 \\ 6-6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

There are two algebraic methods to find the inverse of a matrix and one method using technology. All three methods will be discussed here to find the inverse of the matrix $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$.

Method 1: System of Equations

Using the property that $AA^{-1} = I$, let $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, and then:

$$AA^{-1} = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Multiply the matrices together:

$$\begin{bmatrix} 2w + 6y & 2x + 6z \\ w + 4y & x + 4z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix equality creates the following system of equations:

$$\begin{aligned} 2w + 6y &= 1 & \qquad 2x + 6z &= 0 \\ w + 4y &= 0 & \qquad x + 4z &= 1 \end{aligned}$$

Solve the left system for w and y using the elimination method:

$$\begin{aligned} 2w + 6y = 1 \quad &\rightarrow \quad 2w + 6y = 1 \\ -2(w + 4y = 0) &\rightarrow -2w - 8y = 0 \end{aligned}$$

Add the equations together:

$$\begin{aligned} -2y &= 1 \\ y &= -\frac{1}{2} \end{aligned}$$

Substitute the value for y into one of the two original equations:

$$\begin{aligned} w + 4\left(-\frac{1}{2}\right) &= 0 \\ w - 2 &= 0 \\ w &= 2 \end{aligned}$$

Solve the right system for x and z using the elimination method:

$$\begin{aligned} 2x + 6z = 0 \quad &\rightarrow \quad 2x + 6z = 0 \\ -2(x + 4z = 1) &\rightarrow -2x - 8z = -2 \end{aligned}$$

Add the equations together:

$$\begin{aligned} -2z &= -2 \\ z &= 1 \end{aligned}$$

Substitute the value for z into one of the two original equations:

$$x + 4(1) = 1$$

$$x + 4 = 1$$

$$x = -3$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Method 2: The Determinant

The determinant of the 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$ and is notated det(A). The value of the determinant can be used to find the inverse of a matrix. If det(A) ≠ 0, then if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The determinant of $\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ is (2)(4) − (6)(1) = 8 − 6 = 2.
Then A^{-1}:

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

> The square matrix A has an inverse if and only if det(A) ≠ 0.

Method 3: Technology

The inverse of a matrix can be found using a graphing calculator. First enter the elements of the matrix by pressing 2ND X^{-1} and moving the right arrow twice to EDIT. Next enter the dimensions, row × column = 2 × 2. Then enter each element of matrix $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$. Type 2ND MODE to return to the home screen, and then press 2ND X^{-1} and ENTER. [A] should appear on the screen. Press the X^{-1} button, and press ENTER. The inverse of the matrix will appear on the screen:

$$\begin{bmatrix} 2 & -3 \\ -.5 & 1 \end{bmatrix}$$

This confirms the results of the two previous methods.

Example

Find the inverse of matrix $A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$.

Solution

Using Method 2, det(A) = (3)(1) − (− 2)(−1) = 1. Therefore, $A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

This can be verified using the calculator.

Application of the Determinant

Matrices can be a useful tool when working with vectors. Vectors $\vec{v_1} = ai + cj = \langle a, c \rangle = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\vec{v_2} = bi + dj = \langle b, d \rangle = \begin{pmatrix} b \\ d \end{pmatrix}$ are shown below in Figure 12.3.

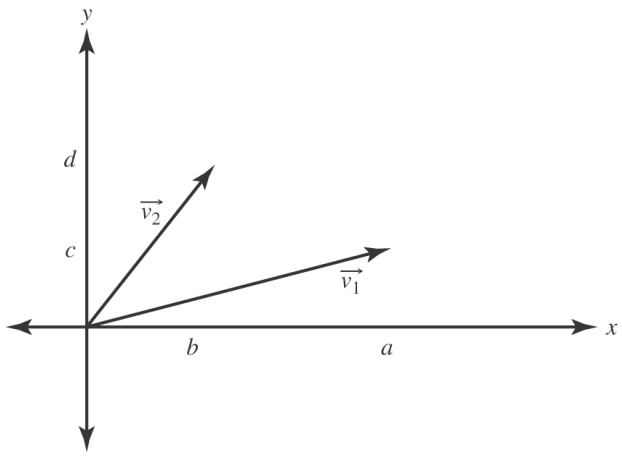


Figure 12.3

The coordinates of the head of $\vec{v_1}$ are (a, c), and the coordinates of the head of $\vec{v_2}$ are (b, d). A parallelogram can be created with the two vectors where their tails both intersect at the origin as shown in Figure 12.4.

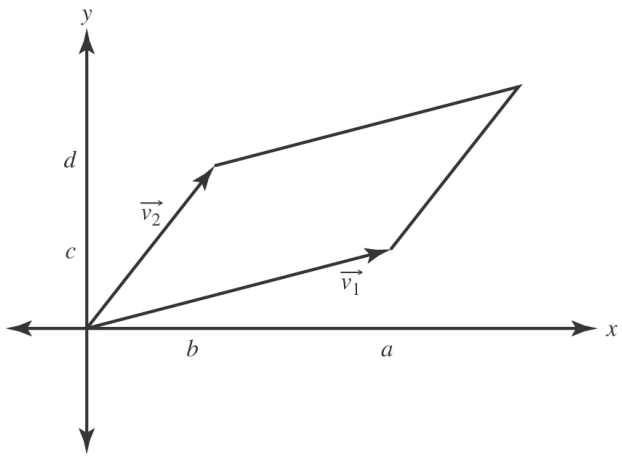


Figure 12.4

Note that the matrix comprised of the components of the vectors can be written as two column vectors or as two row vectors. The area of the parallelogram will still be = |det(*A*)|.

Let *A* represent the 2 × 2 matrix that consists of vectors $\vec{v_1}$ and $\vec{v_2}$ as two column vectors from R^2, where R^2 represents the collection of ordered pairs in a plane. This means $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The area of the parallelogram created by $\vec{v_1}$ and $\vec{v_2} = |\det(A)|$ as long as det(*A*) ≠ 0. If det(*A*) = 0, the vectors are parallel.

Example

Use determinants to calculate the area of the parallelogram with vertices *C*(0, 0), *A*(7, 2), *L*(12, 11), and *Y*(5, 9).

Solution

Begin by graphing the points of the parallelogram to make a model, as shown in Figure 12.5.

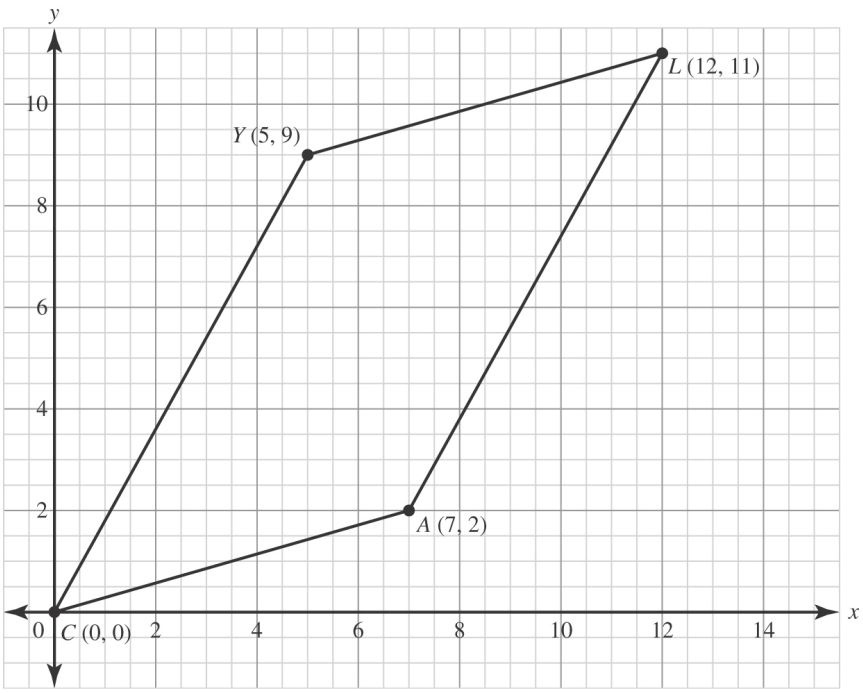


Figure 12.5

Two vectors forming the parallelogram are $\overrightarrow{CA} = \langle 7 - 0, 2 - 0 \rangle = 7\vec{i} + 2\vec{j}$ and $\overrightarrow{CY} = \langle 5 - 0, 9 - 0 \rangle = 5\vec{i} + 9\vec{j}$. Let $E = \begin{bmatrix} 7 & 2 \\ 5 & 9 \end{bmatrix}$, a 2 × 2 matrix that consists of vectors $\overrightarrow{CA}$ and $\overrightarrow{CY}$ as two row vectors from R^2. Calculate the area of the parallelogram:

$$|\det(E)| = |(7)(9) - (2)(5)| = |63 - 10| = |53| = 53 \text{ square units}$$

Example

Use determinants to calculate the area of the parallelogram with vertices (1, 2), (− 3, 4), and (2, 4).

Solution

A graph of the given points is provided in Figure 12.6. It is a visual of the vectors that form the parallelogram.

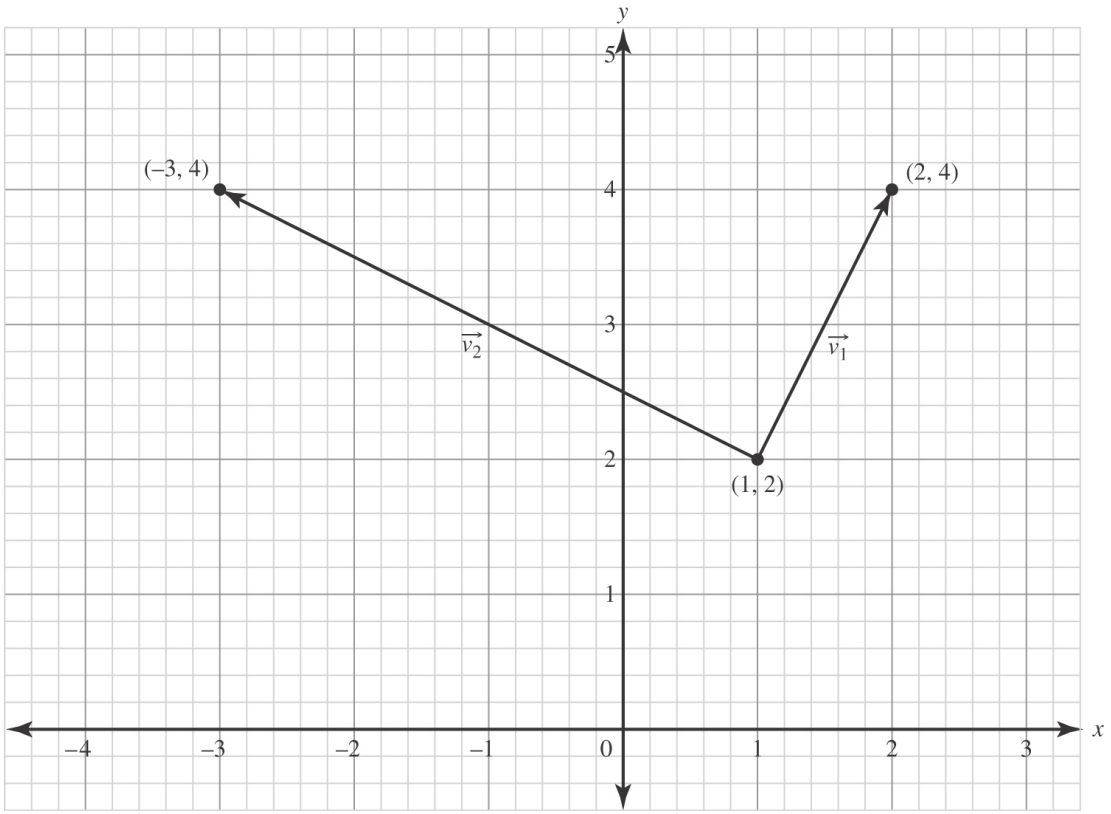


Figure 12.6

There is no need for the fourth coordinate to be provided since it could be calculated using slopes. However, that is not necessary since the three coordinates provided graph the two vectors that form the parallelogram. The two vectors forming the parallelogram are $\vec{v_1} = \langle 2 - 1, 4 - 2 \rangle = 1\vec{i} + 2\vec{j}$ and $\vec{v_2} = \langle -3 - 1, 4 - 2 \rangle = -4\vec{i} + 2\vec{j}$. Let $E = \begin{bmatrix} -4 & 1 \\ 2 & 2 \end{bmatrix}$, a 2 × 2 matrix that consists of vectors $\vec{v_1}$ and $\vec{v_2}$ as two column vectors from R^2. Calculate the area of the parallelogram:

$$|\det(E)| = |(-4)(2) - (1)(2)| = |-8 - 2| = |-10| = 10 \text{ square units}$$

12.3 Linear Transformations and Matrices

Recall that a function is a rule *f* that associates each element in set *A* with one and only one element in set *B*. If *f* associates the element *b* with element *a*, it is notated as *f*(*a*) = *b*. That means *f*(*a*) is the value of *f* at *a*. For

the most common functions, A and B are sets of real numbers, in which case f is called a real-valued function of a real variable.

Other common functions occur when B is a set of real numbers and A is a set of vectors in R^2, R^3, or more generally, in R^n.

When $n = 2$ or $n = 3$, it is common to use the terms ordered pair and ordered triple, respectively, instead of saying ordered 2-tuple and 3-tuple. When $n = 1$, each ordered n-tuple consists of one real number. So R^1 may be viewed as the set of real numbers. It is more common to write R or $\mathbb{R}$ rather than R^1 for this set.

We are most familiar with the study of 2-space. In 2-space, the symbol (a_1, a_2) has two different geometric interpretations. First, it can be interpreted as a point, in which case a_1 and a_2 are the coordinates as shown in Figure 12.7. Second, it can be interpreted as a vector, in which case a_1 and a_2 are the components as shown in Figure 12.8.

If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is called n-space and is denoted by R^n.

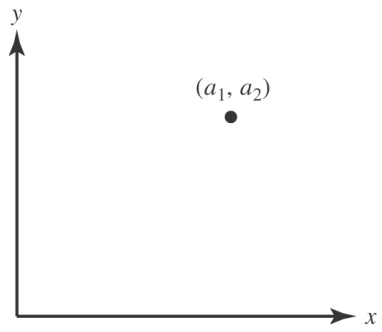


Figure 12.7

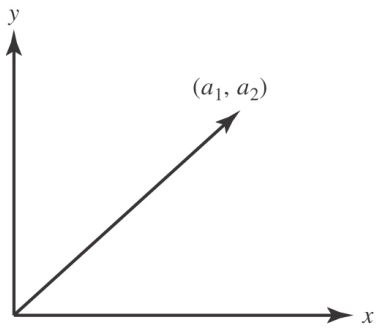


Figure 12.8

It follows, therefore, that an ordered n-tuple $(a_1, a_2, \ldots, a_n)$ can be viewed as a generalized point or as a generalized vector.

Now that we have a better understanding of R^n, we can return to the discussion of common functions in R^n. Table 12.2 outlines some examples.

Table 12.2

Formula	Example	Classification	Description
$f(x)$	$f(x) = x^2$	Real-valued function of a real variable	Function from R to R
$f(x, y)$	$f(x, y) = x^2 + y^2$	Real-valued function of two real variables	Function from R^2 to R
$f(x, y, z)$	$f(x, y, z) = x^2 + y^2 + z^2$	Real-valued function of three real variables	Function from R^3 to R
$f(x_1, x_2, \ldots, x_n)$	$f(x_1, x_2, \ldots, x_n) = {x_1}^2 + {x_2}^2 + {x_3}^2 + \ldots + {x_n}^2$	Real-valued function of n real variables	Function from R^n to R

Functions from R^n to R^m

If the domain of a function f is R^n and the range is R^m, where m and n could be equal, then f is called a map or transformation from R^n to R^m. This is notated as $f : R^n \rightarrow R^m$. The functions shown in Table 12.2 are transformations for which $m = 1$. In the case where $m = n$, the transformation $f : R^n \rightarrow R^n$ is called an operator on R^n. The first entry in Table 12.2 is an operator on R.

To demonstrate one way in which transformations can arise, suppose that $f_1, f_2, \ldots, f_m$ are real-valued functions of n real variables such that:

$$\begin{aligned} w_1 &= f_1(x_1, x_2, \ldots, x_n) \\ w_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \qquad\qquad \vdots \\ w_m &= f_m(x_1, x_2, \ldots, x_n) \end{aligned}$$

These m equations assign a unique point $(w_1, w_2, \ldots, w_m)$ in R^m to each point $(x_1, x_2, \ldots, x_n)$ in R^n and so defines a transformation from R^n to R^m. This transformation can be notated by T and so $T : R^n \rightarrow R^m$ and $T(x_1, x_2, \ldots, x_n) = (w_1, w_2, \ldots, w_m)$.

Example

The equations below define a transformation $T : R^2 \rightarrow R^3$:

$$\begin{aligned} w_1 &= x_1 + x_2 \\ w_2 &= 3x_1 x_2 \\ w_3 &= x_1^2 - x_2^2 \end{aligned}$$

Find the image of the point $(1, -2)$.

Solution

In general, with this transformation the image of the point (x_1, x_2) is

$$T(x_1, x_2) = (x_1 + x_2, 3x_1x_2, {x_1}^2 - {x_2}^2)$$

Therefore:

$$T(1, -2) = (1 + -2, 3(1)(-2), (1)^2 - (-2)^2) = (-1, -6, -3)$$

Linear Transformations from R^n to R^m

In the special case where the following equations are linear:

$$\begin{aligned} w_1 &= f_1(x_1, x_2, \ldots, x_n) \\ w_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \qquad\qquad \vdots \\ w_m &= f_m(x_1, x_2, \ldots, x_n) \end{aligned}$$

The transformation $T : R^n \rightarrow R^m$ defined by those equations is called a linear transformation, or a linear operator if $m = n$. Thus, a linear transformation $T : R^n \rightarrow R^m$ is defined by equations of the following form:

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \qquad\qquad \vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

or is written in matrix notation:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or can be written more generally as:

$$\vec{w} = A\vec{x}$$

The following matrix is called the standard matrix for the linear transformation T, and T is called multiplication by A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If $T : R^n \rightarrow R^m$ is multiplication by A, this can also be notated as $T_A : R^n \rightarrow R^m$ or $T_A(\vec{x}) = A\vec{x}$, where the vector $\vec{x}$ in R^n is expressed as a column matrix.

Geometric Effect of an Operator

Note that depending on whether the inputs are regarded as points or vectors, the geometric effect of an operator $T : R^n \rightarrow R^m$ is to transform each point or vector in R n into some new point or vector. This is shown in Figures 12.9 and 12.10.

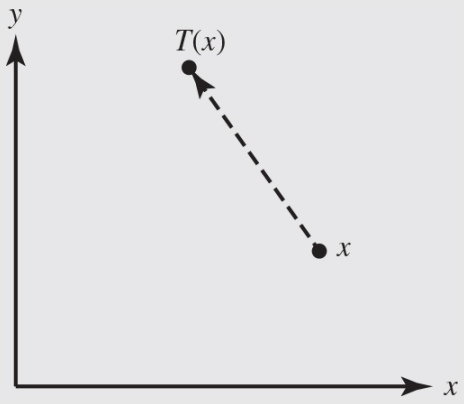


Figure 12.9

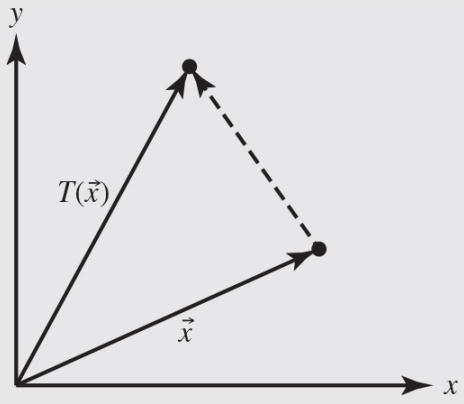


Figure 12.10

Let A be a 2 × 3 matrix such that $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix}$. If A is multiplied by the vector $\vec{v} = \langle x, y, z \rangle$ then:

$$A\vec{v} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1)(x) + (0)(y) + (-1)(z) \\ (3)(x) + (1)(y) + (2)(z) \end{bmatrix} = \begin{bmatrix} x - z \\ 3x + y + 2z \end{bmatrix} = \langle x - z,\ 3x + y + 2z \rangle$$

What was created was a function of three variables, (x, y, z), whose output is a two-dimensional vector $x - z,\ 3x + y + 2z$ that can be defined using function notation as $f = A\vec{v}$ where $f : R^3 \rightarrow R^2$. For example, $f(1, 2, 3) = (1) - (3),\ 3(1) + (2) + 2(3) = -2,\ 11$.

Given any $m \times n$ matrix B, a function can be defined such that $g : R^n \rightarrow R^m$ by $g(\vec{x}) = B\vec{x}$, where $\vec{x}$ is an n-dimensional vector.

Example

Find the image of vector $\vec{x} = \langle 1, -3, 0, 2 \rangle$ under the linear transformation $T : R^4 \rightarrow R^3$ defined by the following equations:

$$w_1 = 2x_1 - 3x_2 + x_3 - 5x_4$$

$$w_2 = 4x_1 + x_2 - 2x_3 + x_4$$

$$w_3 = 5x_1 - 2x_2 + 4x_3$$

Solution

The image vector can be expressed in matrix form as:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} (2)(1) + (-3)(-3) + (1)(0) + (-5)(2) \\ (4)(1) + (1)(-3) + (-2)(0) + (1)(2) \\ (5)(1) + (-2)(-3) + (4)(0) + (0)(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix}$$

A linear transformation is a function that maps an input vector to an output vector such that each component of the output vector is the sum of constant multiples of the input vector components.

The function $g(\vec{x})$ is a linear transformation if each term of each component of $g(\vec{x})$ is a number multiplied by one of the variables. For example, the functions $f(x,y) = \left(2x + y, \frac{y}{2}\right)$ and $g(x, y, z) = (z, 0, 1.2x)$ are linear transformations. However, none of the following functions are linear transformations: $h(x, y) = (x^2, y, x)$; $m(x, y, z) = (y, xyz)$; or $n(x, y, z) = (x + 1, y, z)$.

Zero Transformation from R^n to R^m

If 0 is the $m \times n$ zero matrix and $\vec{0}$ is the zero vector in R^n, for every vector $\vec{x}$ in R^n, $0\vec{x} = \vec{0}$. In other words, multiplication by 0 maps every vector in R^n into the zero vector in R^m.

A linear transformation will map the zero vector to the zero vector.

12.4 Matrices as Functions

In the previous section, it was seen that the linear transformation that maps the vector x, y to $a_{11}\,x + a_{12}\,y, a_{21}\,x + a_{22}\,y$ is associated with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

In this way, every matrix can be associated with a function. Going in the opposite direction, a function can be associated with a matrix only if the function is a linear transformation.

› Example

Find the matrix A associated with the function $f(x, y) = (2x + y, y, x - 3y)$, which is a linear transformation from R^2 to R^3.

✓ Solution

The matrix A associated with f is a 3×2 matrix that can be written in general as:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

> The mapping of the unit vectors in a linear transformation provides valuable information for determining the associated matrix.

Matrix A needs to satisfy $f(\vec{x}) = A\vec{x}$, where $\vec{x} = \langle x, y \rangle$. To aid in finding matrix A, use the unit vectors $1, 0$ and $0, 1$.

Let $\vec{x} = \langle 1, 0 \rangle$. Then $f(\vec{x}) = A\vec{x}$ is the first column of A due to matrix multiplication. Therefore, the first column of A is

$$f(1, 0) = (2(1) + (0), (0), (1) - 3(0)) = (2, 0, 1) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Similarly, if $\vec{x} = \langle 0, 1 \rangle$, then $f(\vec{x}) = A\vec{x}$ is the second column of A. This can be shown as:

$$f(0, 1) = (2(0) + (1), (1), (0) - 3(1)) = (1, 1, -3) = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

Putting these together shows that the linear transformation $f(\vec{x})$ is associated with the following matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & -3 \end{bmatrix}$$

Rotation Operators

A linear operator that rotates each vector in R^2 with an initial point at (0, 0) through a fixed angle θ is called a rotation operator on R^2. This is shown in Figure 12.11.

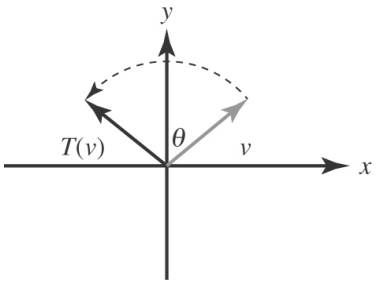


Figure 12.11

It can be shown that the standard matrix $[T]$ of the rotation operator $T : R^2 \rightarrow R^2$ that rotates each vector counterclockwise about the origin through an angle of θ radians is

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Therefore, the image $\vec{w}$ of vector $\vec{x} = \langle x, y \rangle$ can be found by:

$$\vec{w} = \langle w_1, w_2 \rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \langle x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta \rangle$$

Example

Find the image of vector $\langle 1, 1 \rangle$ in R^2 rotated through an angle of $\frac{\pi}{6}$.

Solution

$$\vec{w} = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix}$$

Dilations

Let the function $T : R^2 \rightarrow R^2$ be a two-dimensional linear transformation of the form $T(\vec{x}) = A\vec{x}$ where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. T maps objects from the xy-plane onto an $x'y'$-plane: $(x', y') = T(x, y)$. It will be shown that the geometric properties of this mapping will be reflected in the determinant of the matrix A associated with T.

Consider this linear transformation:

$$T(x, y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

To understand the behavior of T, consider how it maps the unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$.

For the unit vector $\langle 1, 0 \rangle$:

$$T(1, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \langle -2, 0 \rangle$$

For the unit vector $\langle 0, 1 \rangle$:

$$T(0, 1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \langle 0, -2 \rangle$$

The vectors were both stretched by a factor of 2, and the change in the sign of the components occurred from a rotation of π radians.

Figure 12.12 shows the mapping of T on the unit square $[0, 1] \times [1, 0]$, demonstrating that T did rotate the square by π radians around the origin and stretched each side of the square by a factor of 2.

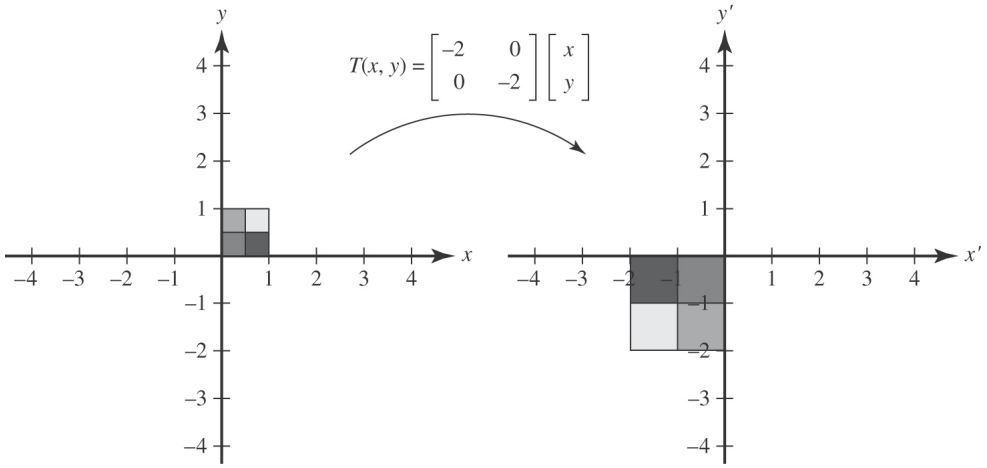


Figure 12.12

The area of the original unit square is (1)(1) = 1 square unit. The area of the transformed square is (2)(2) = 4 square units.

The absolute value of the determinant of a 2 × 2 transformation matrix gives the magnitude of the dilation of regions in R^2 under the transformation.

In Section 12.2, it was discovered that the absolute value of the determinant of a 2 × 2 matrix is the area of the parallelogram spanned by the vectors represented in the columns or rows of the matrix. In this case, the area of the square, which is a parallelogram, is 4 and the determinant of the matrix A associated with T is $\det(A) = (-2)(-2) - (0)(0) = 4$.

Example

Given the linear transformation $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$, find the magnitude of the dilation of regions in R^2 under the transformation.

Solution

Calculate the absolute value of the determinant:

$$A = |\det(A)| = |(-1)(3) - (-1)(1)| = |-3 - (-1)| = |-2| = 2$$

Compositions of Linear Transformations

If T_A: $R^n \rightarrow R^k$ and T_B: $R^k \rightarrow R^m$ are linear transformations, for each $\vec{x}$ in R^n you can first compute $T_A(\vec{x})$, which is a vector in R^k, and then compute $T_B(T_A(\vec{x}))$, which is a vector in R^m. Therefore, the composition of T_B with T_A denoted as $(T_B \circ T_A)(\vec{x}) = T_B(T_A(\vec{x}))$ produces a transformation from R^n to R^m.

The composition $T_B \circ T_A$ is linear since:

$$(T_B \circ T_A)(\vec{x}) = T_B(T_A(\vec{x})) = B(A\vec{x}) = (BA)\vec{x}$$

So $T_B \circ T_A$ is multiplication by the matrices BA, which is a linear transformation.

This leads to two big discoveries outlined below.

Two Big Discoveries

1. The composition of two linear transformations is a linear transformation.
2. The matrix associated with the composition of two linear transformations is the product of the matrices associated with each linear transformation.

› Example

Given the following linear transformations:

$$T: R^2 \to R^2 \text{ by } T(\vec{u}) = \begin{bmatrix} u_1 + u_2 \\ 3u_1 + 3u_2 \end{bmatrix}$$

$$S: R^2 \to R^2 \text{ by } T(\vec{v}) = \begin{bmatrix} 3v_1 - v_2 \\ -3v_1 + v_2 \end{bmatrix}$$

express $S \circ T$ as a matrix transformation.

✓ Solution

The matrix associated with the composition of the given linear transformations is the product of the matrices associated with each linear transformation.

The matrix associated with T is $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$, and the matrix associated with S is $\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$ is

$$\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} (3)(1) + (-1)(3) & (3)(1) + (-1)(3) \\ (-3)(1) + (1)(3) & (-3)(1) + (1)(3) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This means that $S \circ T$ maps all vectors of R^2 to the zero vector.

Inverse of a Linear Transformation

Inverses of linear transformations are like inverses of functions. If A is an $n \times n$ matrix and if $T_A : R^n \rightarrow R^n$ is multiplication by A, the following statements are equivalent:

- A is invertible.
- The range of T_A is R^n.
- T_A is one-to-one.

If $T_A : R^n \rightarrow R^n$ is a one-to-one linear operator, the matrix A is invertible. So $T_A -1 : R^n \rightarrow R^n$ is also a linear operator and is called the inverse of T_A. The linear operators T_A and $T_A -1$ cancel the effect of one another, which means:

$$T_A\left(T_{A^{-1}}(\vec{x})\right) = AA^{-1}\vec{x} = I\vec{x} = \vec{x}$$

$$T_{A^{-1}}\left(T_A(\vec{x})\right) = A^{-1}A\vec{x} = I\vec{x} = \vec{x}$$

From a more geometric viewpoint, if $\vec{w}$ is the image of $\vec{x}$ under T_A, then $T_{A^{-1}}$ maps $\vec{w}$ back into $\vec{x}$ since:

$$T_{A^{-1}}(\vec{w}) = T_{A^{-1}}\left(T_A(\vec{x})\right) = \vec{x}$$

This is shown in Figure 12.13.

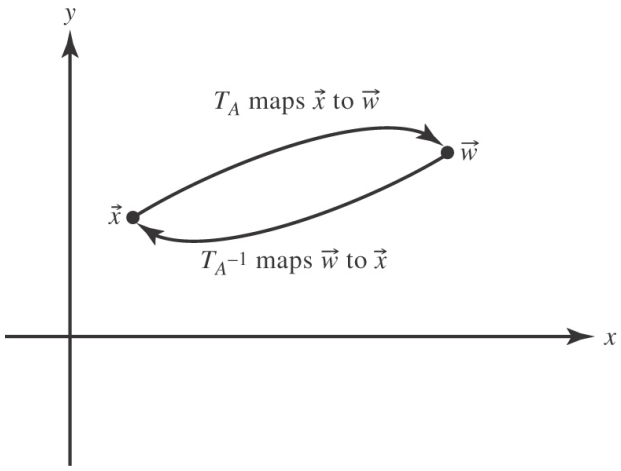


Figure 12.13

When a one-to-one linear operator on R^n is written as $T : R^n \rightarrow R^n$ rather than $T_A : R^n \rightarrow R^n$, the inverse operator T is denoted by T^{-1} rather than ${T_A}^{-1}$.

Example

Show that the linear operator $T : R^2 \rightarrow R^2$ defined by the following equations:

$$w_1 = 2x_1 + x_2$$
$$w_2 = 3x_1 + 4x_2$$

is one-to-one, and find $T^{-1}(w_1, w_2)$.

Solution

The matrix form is written as:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So the standard matrix for T is

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

The determinant of this matrix is (2)(4) − (1)(3) = 8 − 3 = 5. Since the determinant is not 0, the matrix is invertible. Therefore, T is one-to-one.

The standard matrix for T^{-1} is

$$\frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

Therefore:

$$T^{-1}(\vec{w}) = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5}w_1 - \frac{1}{5}w_2 \\ -\frac{3}{5}w_1 + \frac{2}{5}w_2 \end{bmatrix}$$

This means:

$$T^{-1}(w_1, w_2) = \left(\frac{4}{5}w_1 - \frac{1}{5}w_2, -\frac{3}{5}w_1 + \frac{2}{5}w_2\right)$$

12.5 Matrices Modeling Contexts

Suppose a physical or mathematical system undergoes a process of change such that at any moment it can occupy one of a finite number of states. For example, the weather in a certain city could be in one of four possible states: sunny, cloudy, rainy, or snowy. Suppose that such a system changes with time from one state to another and that at scheduled times the state of the system is observed. If the state of the system at any observation cannot be predicted by just knowing the state of the system at the preceding observation, the process of change is called a Markov chain.

A Markov Chain

If a Markov chain has k possible states, which are labeled as 1, 2, …, k, the probability that the system is in state i at any observation after it was in state j at the preceding observation is denoted by p_{ij} and is called the transition probability from state j

to state i. The matrix $P = [p_{ij}]$ is called the transition matrix of the Markov chain.

For example, in a three-state Markov chain, the transition matrix has the form shown in Figure 12.14.

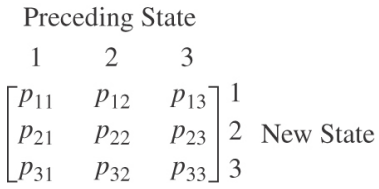


Figure 12.14

In this matrix, p_{32} is the probability that the system will change from state 2 to state 3, p_{11} is the probability that the system will remain in state 1 if it was previously in state 1, etc.

Example

A library has three locations denoted by 1, 2, and 3. A patron may borrow a book from any of the three locations and return the book to any of the three locations. The head librarian finds that patrons return the books to the various locations according to the probabilities found in Figure 12.15.

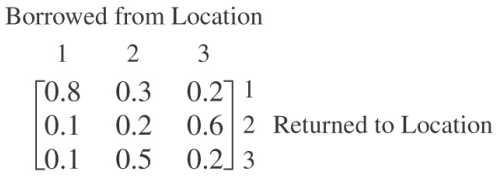


Figure 12.15

1. What is the probability that a book borrowed from location 3 will be returned to location 2?
2. Which scenario has the highest probability?

✓ Solution

The given matrix is the transition matrix of the Markov chain. From this matrix, the probabilities of the different scenarios of a patron borrowing and returning books to the library locations can be determined.

1. The probability is 0.6 that a book borrowed from location 3 will be returned to location 2.
2. The highest probability is 0.8. This is the probability that a book borrowed from location 1 will be returned to location 1.

A Markov Matrix

In the previous example, the transition matrix of the Markov chain has the property that the entries in any column add to 1. If $P = [p_{ij}]$ is the transition matrix of any Markov chain with k states, for each j it is true that $p_{1j} + p_{2j} + \ldots + p_{kj} = 1$. A matrix with this property is known as a Markov matrix.

Matrices of the Markov chain are to be used to make predictions about future and past states. Although in a Markov chain the state of the system at any observation time cannot generally be determined with certainty, the best that can be done is to specify probabilities for each of the possible states. For example, in a Markov chain with 3 states, the possible state of the system at some observation time can be described by a column vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where x_1 is the probability that the system is in state 1, x_2 is the probability that it is in state 2, and x_3 is the probability that it is in state 3.

Knowing the initial state vector $\vec{x}^{(0)}$ for a Markov chain makes it possible to determine the state vectors $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}, \ldots$ at future observation times.

The state vector for an observation of a Markov chain with k states is a column vector $\vec{x}$ whose ith component x_i is the probability that the system is in the ith state at that time.

The entries in any state vector for a Markov chain are nonnegative and have a sum of 1.

If P is the transition matrix of a Markov chain and $\vec{x}^{(n)}$ is the state vector at the nth observation, $\vec{x}^{(n+1)} = P\vec{x}^{(n)}$.

Example

By reviewing its donation records, the alumni office of a college finds that 80% of its alumni who contribute to the annual fund one year will also contribute the next year and that 30% of those who do not contribute one year will contribute the next.

(a) Create a transition matrix for this Markov chain with two states. State 1 corresponds to an alumnus giving a donation in any one year. State 2 corresponds to the alumnus not giving a donation in that year.

(b) Construct the probable future donation record for the next 3 years of a new graduate who did not give a donation in the initial year after graduation.

(c) Using technology, construct the probable future donation record for the next 9, 10, and 11 years for the person described in part (b).

Solution

(a) State 1 corresponds to an alumnus giving a donation in any one year, and state 2 corresponds to the alumnus not giving a donation in that year. So the transition matrix is

$$P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

(b) For a graduate who did not give a donation in the initial year after graduation, the system is initially in state 2 with certainty. So the initial state vector is

$$\vec{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To predict that person's future donation records, perform repeated multiplication of the transition matrix and the corresponding resultant state vectors:

$$\vec{x}^{(1)} = P\vec{x}^{(0)} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$$

$$\vec{x}^{(2)} = P\vec{x}^{(1)} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix}$$

$$\vec{x}^{(3)} = P\vec{x}^{(2)} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 0.525 \\ 0.475 \end{bmatrix}$$

After 3 years, the alumnus can be expected to make a donation with probability 0.525.

(c) To figure out the probable future donation record for the next 9, 10, and 11 years, the multiplication shown in part (b) will be repeated. So using a graphing calculator can prove useful:

$$\vec{x}^{(9)} = P\vec{x}^{(8)} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.598 \\ 0.402 \end{bmatrix} = \begin{bmatrix} 0.599 \\ 0.401 \end{bmatrix}$$

$$\vec{x}^{(10)} = P\vec{x}^{(9)} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.599 \\ 0.401 \end{bmatrix} = \begin{bmatrix} 0.599 \\ 0.401 \end{bmatrix}$$

$$\vec{x}^{(11)} = P\vec{x}^{(10)} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.599 \\ 0.401 \end{bmatrix} = \begin{bmatrix} 0.600 \\ 0.400 \end{bmatrix}$$

Repeated multiplication of a matrix that models the transitions between states and corresponding resultant state vectors can predict the steady state, which is a distribution between states that does not change from one step to the next.

After 9 years, the alumnus can be expected to make a donation with probability 0.599. After 10 years, the alumnus can be expected to make a donation with probability 0.599. After 11 years, the alumnus can be expected to make a donation with probability 0.600. In this example, it can be observed that the state vectors converge to a fixed vector as the number of observations increases.

A similar procedure can be performed to predict past states. The product of the inverse of a matrix that models transitions between states and a corresponding state vector can predict past states.

› Example

Use the inverse of the transition matrix to find the state vector that represents the future donation record of the alumnus after 7 years as described in the scenario in the previous example.

✓ Solution

Since $\vec{x}^{(8)} = \begin{bmatrix} 0.598 \\ 0.402 \end{bmatrix}$ and the transition matrix $P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$, the determinant can be found:

$$\det(P) = (0.8)(0.7) - (0.3)(0.2) = 0.5$$

Find the inverse of the transition matrix:

$$P^{-1} = \frac{1}{0.5}\begin{bmatrix} 0.7 & -0.3 \\ -0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 1.4 & -0.6 \\ -0.4 & 1.6 \end{bmatrix}$$

Therefore:

$$\vec{x}^{(7)} = P^{-1}\vec{x}^{(8)} = \begin{bmatrix} 1.4 & -0.6 \\ -0.4 & 1.6 \end{bmatrix}\begin{bmatrix} 0.598 \\ 0.402 \end{bmatrix} = \begin{bmatrix} 0.596 \\ 0.404 \end{bmatrix}$$

This answer can be verified from the work completed in part (c) of the previous example.

Multiple-Choice Questions

1. A certain stock price has been observed to follow a pattern. If the stock price goes up one day, there's a 20% chance of it rising tomorrow, a 30% chance of it falling, and a 50% chance of it remaining the same. If the stock price falls one day, there's a 35% chance of it rising tomorrow, a 50% chance of it falling, and a 15% chance of it remaining the same. Finally, if the price is stable on one day, then it has a 50% chance of rising the next day and a 50% chance of falling. Which matrix is the transition matrix for this Markov chain if the states are listed in order of rising, falling, and constant?

(A) $\begin{bmatrix} 20 & 30 & 50 \\ 35 & 50 & 15 \\ 50 & 50 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0.2 & 0.35 & 0.5 \\ 0.3 & 0.5 & 0.5 \\ 0.5 & 0.15 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 20 & 35 & 50 \\ 30 & 50 & 50 \\ 50 & 15 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.35 & 0.5 & 0.15 \\ 0.5 & 0.5 & 0 \end{bmatrix}$

2. Calculate the area of the parallelogram with the given vertices $C(0, 0)$, $M(2, 6)$, $P(11, 8)$, and $J(9, 2)$.

(A) 100

(B) 52

(C) 50

(D) 49

3. Which of the following is the inverse matrix of $A = \begin{bmatrix} -5 & 4 \\ 0 & 4 \end{bmatrix}$?

(A) $\begin{bmatrix} -\frac{1}{5} & -\frac{1}{5} \\ 0 & \frac{1}{4} \end{bmatrix}$

(B) $\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{4} \end{bmatrix}$

(C) $\begin{bmatrix} \frac{1}{4} & \frac{1}{5} \\ 0 & -\frac{1}{5} \end{bmatrix}$

(D) $\begin{bmatrix} 0 & \frac{1}{4} \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$

4. Let $A = [-5 \quad 2]$ and $B = [1 \quad 0]$. Find $2A + 3B$.

(A) $[-10 \quad 4]$
(B) $[-9 \quad 4]$
(C) $[-7 \quad 4]$
(D) $[-2 \quad 2]$

5. Let $A = \begin{bmatrix} 1 & 3 & -3 \\ 3 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ -3 & 1 \\ 0 & 5 \end{bmatrix}$. Find AB if it is defined.

(A) $\begin{bmatrix} -12 & -6 \\ 25 & 9 \end{bmatrix}$

(B) $\begin{bmatrix} -6 & -12 \\ 9 & 25 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & -9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$

(D) AB is undefined

6. Which of the following is the correct geometric interpretation of the associated linear transformations for the standard matrix $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$??

(A) Rotates counterclockwise through π radians
(B) Rotates counterclockwise through $\frac{\pi}{2}$ radians and doubles the length
(C) Rotates counterclockwise through $\frac{\pi}{2}$ radians and halves the length
(D) Rotates counterclockwise through $\frac{\pi}{2}$ radians and quadruples the length

7. Let T be a linear transformation from R^n to R^m and let $\vec{v} = T(\vec{0})$, where $\vec{0}$ is the zero vector in R^n. Which of the following statements is true?

(A) $\vec{v}$ is a zero vector in R^n.

(B) $\vec{v}$ is a zero vector in R^m if and only if $n = m$.

(C) $\vec{v}$ is a zero vector in R^m.

(D) None of the statements is true.

8. Consider the following three statements about an invertible $n \times n$ matrix A.

I. $A = A^{-1}$

II. $AA = I$

III. $A^{-1}A^{-1} = I$

Which of the statements are equivalent?

(A) I and II only

(B) II and III only

(C) I and III only

(D) I, II, and III

9. Suppose $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$. Find A.

(A) $\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$

(B) $\begin{bmatrix} 11 & -17 \\ -27 & 41 \end{bmatrix}$

(C) $\begin{bmatrix} 31 & 43 \\ 4 & 18 \end{bmatrix}$

(D) $\begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$

10. Given that $T : R^3 \to R^2$, which of the following choices is not a linear transformation?

(A) $T(x, y, z) = (x, x + y + z)$

(B) $T(x, y, z) = (1, 1)$

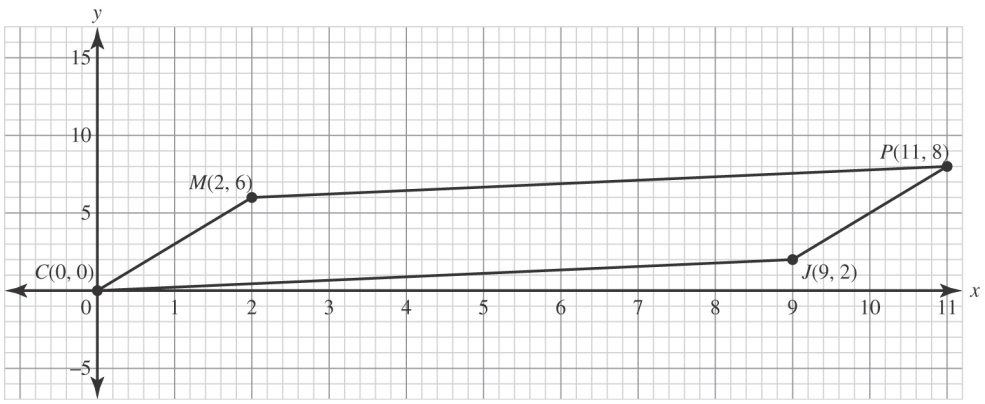
(C) $T(x, y, z) = (0, 0)$

(D) $T(x, y, z) = (3x - 4y, 2x - 5z)$

Answer Explanations

1. **(B)** The transition matrix of the Markov chain should have columns that sum to 1. This eliminates Choices (A), (C), and (D). The transition matrix in Choice (B) reflects the correct presentation of the given information for each of the days and the given percentages.

2. **(C)** The graph of the parallelogram is provided below:



The two vectors forming the parallelogram are $\overrightarrow{CM} = \langle 2-0, 6-0\rangle = 2\vec{i} + 6\vec{j}$ and $\overrightarrow{CJ} = \langle 9-0, 2-0\rangle = 9\vec{i} + 2\vec{j}$. Let $E = \begin{bmatrix} 2 & 6 \\ 9 & 2 \end{bmatrix}$, a 2×2, a 2×2 matrix that consists of vectors $\overrightarrow{CM}$ and $\overrightarrow{CJ}$ as two row vectors from R^2. The area of the parallelogram can be calculated as follows:

$$|\det(E)| = |(2)(2) - (6)(9)| = |4 - 54| = |-50| = 50 \text{ square units}$$

3. **(B)** The $\det(A) = (-5)(4) - (4)(0) = -20$. Since $\det(A) \neq 0$, matrix A is invertible:

$$A^{-1} = -\frac{1}{20}\begin{bmatrix} 4 & -4 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{4} \end{bmatrix}$$

4. **(C)** $2A + 3B = 2[-\quad 5^2] + 3[1 \quad 0] = [-10 \quad 4] + [3 \quad 0] = [-7 \quad 4]$.

5. **(B)** Since A is a 2×3 matrix and B is a 3×2 matrix, they can be multiplied together. Their product will be a 2×2 matrix. The product is AB:

$$AB = \begin{bmatrix} 1 & 3 & -3 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -3 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} (1)(3)+(3)(-3)+(-3)(0) & (1)(0)+(3)(1)+(-3)(5) \\ (3)(3)+(0)(-3)+(5)(0) & (3)(0)+(0)(1)+(5)(5) \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ 9 & 25 \end{bmatrix}$$

6. **(B)** Let the function $T : R^2 \rightarrow R^2$ be a two-dimensional linear transformation of the form $T(\vec{x}) = A\vec{x}$ where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The linear transformation given was that of $T(x, y) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. To understand the behavior of *T*, consider how it maps the unit vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$.

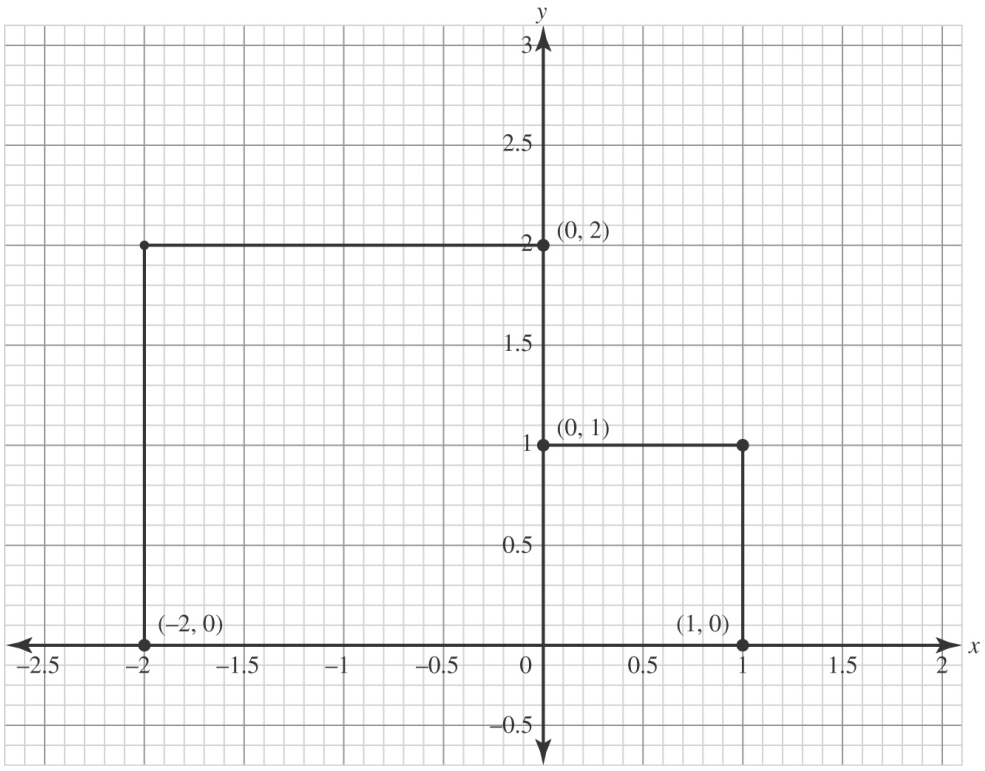
For the unit vector $\langle 1, 0 \rangle$:

$$T(1, 0) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \langle 0, 2 \rangle$$

For the unit vector $\langle 0, 1 \rangle$:

$$T(0, 1) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \langle -2, 0 \rangle$$

The transformation can be illustrated graphically.



Practice Tests

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Practice Test 1

Section I

Part A: Graphing calculator not permitted

TIME: 80 MINUTES

DIRECTIONS: For each question or incomplete statement, select the choice that best answers the question or completes the statement.

1. What is the numerical coefficient of the term containing $x^3 y^2$ in the expansion of $(x + 2y)^5$?

   (A) 10
   (B) 20
   (C) 40
   (D) 80

2. If $f(x) = x^3 + Ax^2 + Bx - 3$ and if $f(1) = 4$ and $f(-1) = -6$, what is the value of $2A + B$?

   (A) −2
   (B) 0

(C) 8

(D) 12

3. If $f(x) = x^3 - 2x - 1$, then $f(-2) =$

(A) −17

(B) −13

(C) −5

(D) −1

4. What is the domain of $f(x) = \frac{x-5}{x^2+25}$?

(A) All reals, $x \neq 5$

(B) All reals, $x \neq \pm 5$

(C) All reals, $x \neq -5$

(D) All reals

5. What is the domain of $g(x) = \frac{\sqrt{x-3}}{x^2-2x}$?

(A) All reals, $x \neq 0, 2$

(B) All reals, $x \geq -3$, $x \neq 0, 2$

(C) All reals, $x \geq 3$

(D) All reals

6. Which of the following is equivalent to the expression $\sqrt[3]{\frac{a^2b}{c}}$?

(A) $\sqrt[3]{\frac{2\log a + \log b}{\log c}}$

(B) $\frac{1}{3}\left(\frac{2\log a + \log b}{\log c}\right)$

(C) $\frac{1}{3}\left(\frac{2\log a + \log b}{\log c}\right)$

(D) $\sqrt[3]{2\log a + \log b - \log c}$

7. Identify the vertical asymptote(s) for the function $f(x) = \frac{x^2-4}{x^3+2x^2}$.

(A) $x = 0$

(B) $x = -2, x = 2$

(C) $x = 0, x = 2$

(D) $x = -2, x = 0$

8. Given $f(x) = \begin{cases} -\frac{1}{3}x^2 + 9, x \le 0 \\ (x-3)^2, x > 0 \end{cases}$, find $f(-3)$.

(A) 36

(B) 10

(C) 8

(D) 6

9. Find the slant asymptote of $f(x) = \frac{x^2 + 2x - 1}{x - 1}$.

(A) $y = 1$

(B) $y = x + 3$

(C) $y = x + 1$

(D) $y = x - 1$

10. In polar coordinates, which of the following choices is *not* equivalent to $\left(2, \frac{5\pi}{6}\right)$?

(A) $\left(-2, -\frac{\pi}{6}\right)$

(B) $\left(-2, \frac{11\pi}{6}\right)$

(C) $\left(2, -\frac{11\pi}{6}\right)$

(D) $\left(2, -\frac{7\pi}{6}\right)$

11. Which of the following represents zeros of $r = 2 - 4 \sin 2\theta$?

(A) $\frac{\pi}{6}, \frac{5\pi}{6}$

(B) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

(C) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(D) $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

12. Evaluate: $\cos\left[\arctan\left(-\frac{2}{3}\right)\right]$, given that $\frac{\pi}{2} \le \theta \le \pi$

(A) $-\frac{3\sqrt{13}}{13}$

(B) $-\frac{2\sqrt{13}}{13}$

(C) $\frac{2\sqrt{13}}{13}$

(D) $\frac{3\sqrt{13}}{13}$

13. Given that $\sin\theta = -\frac{1}{5}$ and tan $\theta > 0$, find cos θ.

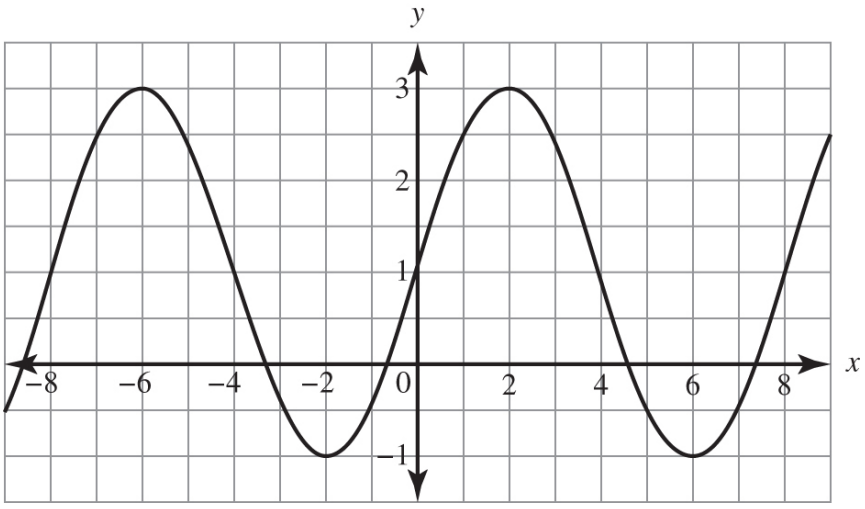
(A) $\frac{\sqrt{26}}{5}$

(B) $\frac{2\sqrt{6}}{5}$

(C) $-\frac{2\sqrt{6}}{5}$

(D) $-\frac{\sqrt{26}}{5}$

14. Which of the following is a possible equation for the sinusoidal graph shown below?



(A) $y = 2 \sin(x) + 1$

(B) $y = 2\cos\left(\frac{\pi}{4}x\right) + 1$

(C) $y = 2\sin\left(\frac{\pi}{4}x + 1\right)$

(D) $y = 2\sin\left(\frac{\pi}{4}x\right) + 1$

15. Which of the following is equivalent to sin 5x cos 3x − cos 5x sin 3x?

(A) cos 8x
(B) sin 8x
(C) cos 2x
(D) sin 2x

16. Given $\sin x = -\frac{1}{8}$ and tan x < 0, find sin 2x.

(A) $-\frac{\sqrt{65}}{32}$
(B) $-\frac{1}{4}$
(C) $-\frac{3\sqrt{7}}{32}$
(D) $\frac{3\sqrt{7}}{32}$

17. Determine the period of the function $y = \frac{1}{2}\sin\left(\frac{1}{3}x - \pi\right)$.

(A) $\frac{1}{2}$
(B) $\frac{2\pi}{3}$
(C) 3π
(D) 6π

18. Solve the equation $\log_a 3 - \log_a b = c$ for b.

(A) $\frac{c}{3a}$
(B) $\frac{3}{a^c}$
(C) $\frac{a}{3c}$
(D) a^3c

19. Which of the following statements is true about the function $g(x) = x^7 + x^3 + \sin x$?

(A) The function is even and symmetric with respect to the origin.

(B) The function is even and symmetric with respect to the *y*-axis.

(C) The function is odd and symmetric with respect to the origin.

(D) The function is odd and symmetric with respect to the *y*-axis.

20. Which of these is an equation for the linear function *f* that satisfies the conditions $f(-3) = -7$ and $f(5) = -11$?

(A) $y + 7 = -\frac{1}{2}(x + 3)$

(B) $y + 11 = \frac{1}{2}(x - 5)$

(C) $y - 7 = -\frac{1}{2}(x - 3)$

(D) $y - 11 = -\frac{1}{2}(x + 5)$

21. Given the functions $f(x) = x^2 - 4$ and $g(x) = \sqrt{x} + 4$, determine the composition $(g \circ f)(x)$.

(A) $x^4 - 8x^2 + 12$

(B) $\sqrt{x^2 - 4} + 4$

(C) $x + 8\sqrt{x} + 12$

(D) $x + 12$

22. Given the function $g(x) = -(12x - 7)^2 (34x + 89)^3$, determine $\lim_{x \to -\infty} g(x)$.

(A) $-\infty$

(B) 0

(C) 1

(D) ∞

23. Evaluate $\lim_{x \to \infty} -5^{-x} - 2$.

(A) $-\infty$

(B) -2

(C) 0

(D) ∞

24. What is the function whose graph is a reflection over the *y*-axis of the graph of $f(x) = 1 - 3^x$?

(A) $g(x) = 1 - 3^{-x}$

(B) $g(x) = 1 + 3^{x}$

(C) $g(x) = 3x - 1$

(D) $g(x) = \log_3(x - 1)$

25. Which of the following functions does *not* have an inverse function?

(A) $y = \sin x \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$

(B) $y = x^3 + 2$

(C) $y = \frac{x^2}{x + 2}$

(D) $y = \frac{1}{2}e^x$

26. The table shows the predicted growth of a particular bacteria after various numbers of hours. Write an explicit formula for the sequence of the number of bacteria.

Hours (n)	1	2	3	4	5
Number of Bacteria (a_n)	19	38	57	76	95

(A) $a_n = 19n + 19$

(B) $a_n = n + 19$

(C) $a_n = \frac{1}{19}n$

(D) $a_n = 19n$

27. What are the points where the graph of the polynomial $f(x) = 7(x - 5)(x + 5)^2$ passes through the x-axis?

(A) $x = 5$ only

(B) $x = -5$ only

(C) $x = -5$ and $x = 5$

(D) Nowhere

28. Evaluate: $\lim_{x \to \infty} \frac{3x^2 - 9x - 4}{5x^2 - 4x + 8}$

(A) Does not exist

(B) 0

(C) $\frac{3}{5}$

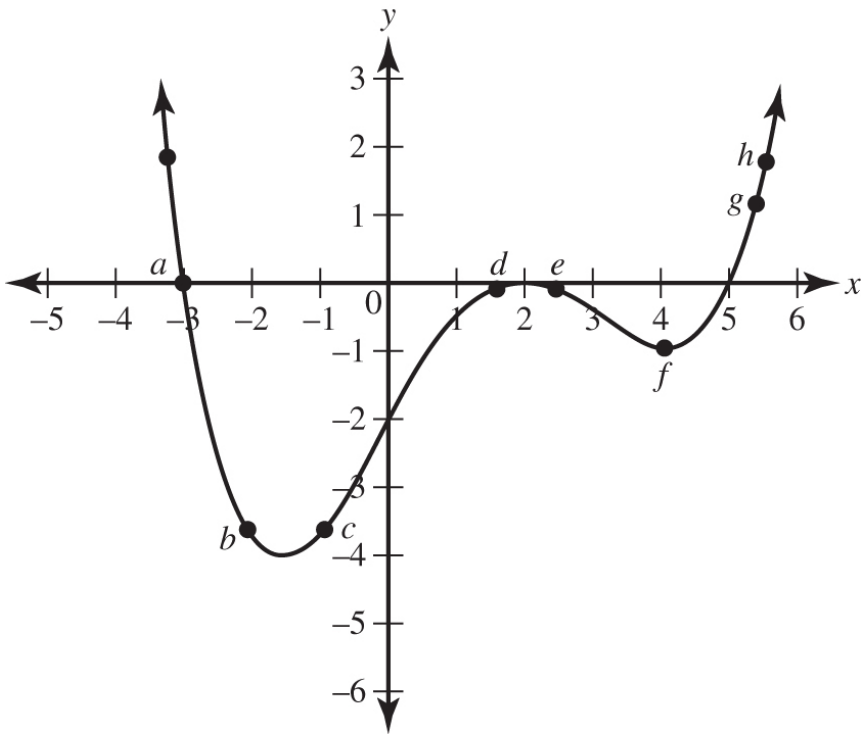
(D) $\frac{9}{4}$

Part B: Graphing calculator required

TIME: 40 MINUTES

DIRECTIONS: For each question or incomplete statement, select the choice that best answers the question or completes the statement.

29. Given the graph of the function $y = f(x)$ shown below with points a, b, c, d, e, f, g, and h, which of the following pairs of points below has the greatest average rate of change?



(A) a and c
(B) b and c
(C) c and d
(D) f and h

30. Given the function $f(x) = (x^2 - 5)(x - 3)(3x - 2)$, at what value of x is the absolute maximum of $f(x)$ over the interval $[-2.28, 3.25]$?

(A) -2.28
(B) -1.27
(C) 1.35
(D) 3.25

31. Approximate $\log_a 24$, given that $\log_a 2 \approx 0.4307$ and $\log_a 3 \approx 0.6826$.

(A) 1.9747
(B) 1.1133
(C) 0.8820
(D) 0.2940

32. Solve for x: $3^{2x} = 5^{x-1}$

(A) -2.7381
(B) -1
(C) -0.5563
(D) 15.2755

33. Which of the following choices is equivalent to the complex number $3 - 8i$?

(A) $\sqrt{73}\left(\cos\frac{37\pi}{60} + i\sin\frac{37\pi}{60}\right)$
(B) $\sqrt{73}\left(\cos\frac{97\pi}{60} + i\sin\frac{97\pi}{60}\right)$
(C) $\sqrt{55}\left(\cos\frac{37\pi}{60} + i\sin\frac{37\pi}{60}\right)$
(D) $\sqrt{55}\left(\cos\frac{97\pi}{60} + i\sin\frac{97\pi}{60}\right)$

34. Find all solutions in the interval $[0, 2\pi]$ for the trigonometric equation $6\cos^2 x - 5\sin x - 2 = 0$.

(A) $-1.333333, -4.4749, \frac{\pi}{6}, \frac{5\pi}{6}$
(B) $\frac{\pi}{6}, \frac{5\pi}{6}$
(C) 2, 5.1416
(D) $\frac{7\pi}{6}, \frac{11\pi}{6}$

35. Give an algebraic expression for $\cos(\sin^{-1} x)$.

(A) $\sqrt{1 - x^2}$
(B) $\sqrt{1 + x^2}$
(C) $\sqrt{x^2 - 1}$
(D) $1 - x^2$

36. How long will it take for $1,000 to double in an investment when interest is compounded continuously at the rate of 5.8% per annum? Round your answer to the nearest year.

(A) 11 years

(B) 12 years

(C) 13 years

(D) 14 years

37. An object is launched straight up from ground level with an initial velocity of 50 feet per second. The height, h (in feet above ground level), of the object t seconds after the launch is given by the function $h(t) = -16t^2 + 50t$. At approximately what value of t will the object have a height of 28 feet and be traveling downward?

(A) 0.73 seconds

(B) 1.56 seconds

(C) 1.84 seconds

(D) 2.39 seconds

38. Suppose you drop a tennis ball from a height of 15 feet. After the ball hits the floor, it rebounds to 85% of its previous height. How high, to the nearest tenth, will the ball rebound after its third bounce?

(A) 1.9 feet

(B) 7.8 feet

(C) 9.2 feet

(D) 10.8 feet

39. Let $f(x) = -2x^6 + 15x^5$. For what values of x does the graph of f have a point of inflection?

I. $x = 0$

II. $x = 5$

III. $x = \frac{15}{2}$

(A) I only

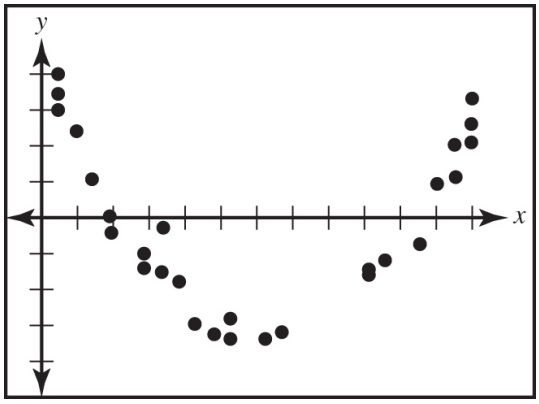
(B) I and II only

(C) I, II, and III

(D) none of these

40. After performing analyses on a set of data, Joseph examined the scatterplot of the residual values for each analysis. Which scatterplot indicates the best linear fit for the data?

(A)



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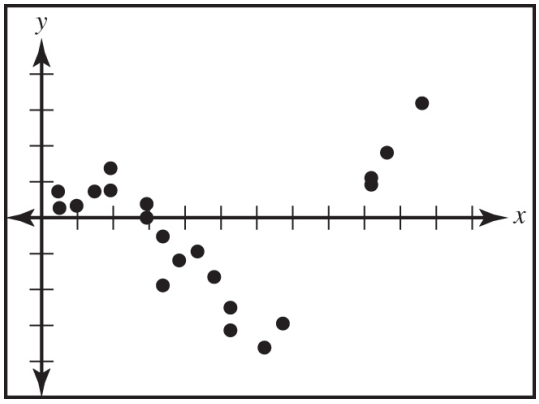
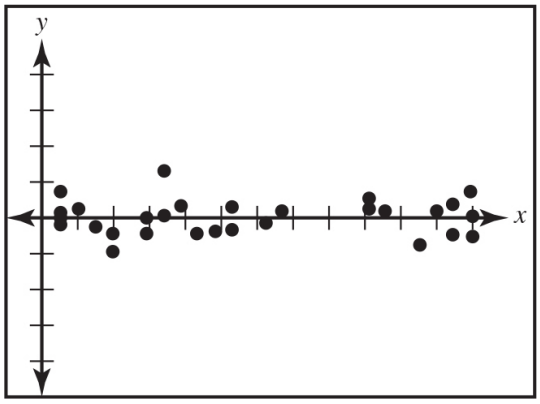
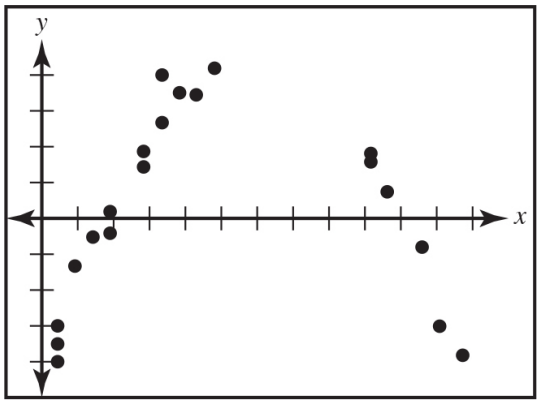


(C)



(D) 

Section II

Part A: Graphing calculator required

TIME: 30 MINUTES

DIRECTIONS: Answer all parts of both free-response questions.

1. A high school decided to host an annual food drive to help local community members during the holiday season. The students recorded how many items were collected at the end of each year's drive. In the initial year, 1980, ($t = 0$), the school collected 50 items. Seven years later ($t = 7$), the school collected 160 items.

The number of items collected on a given year can be modeled by the function F given by $F(t) = a + b \ln (t + 1)$, where $F(t)$ is the number of items collected in a year, and t is the number of years since the initial year of the food drive.

(A) (i) Use the given data to write two equations that can be used to find the values for constants a and b in the expression for $F(t)$.

(ii) Find the values for a and b as decimal approximations, where appropriate.

(B) (i) Use the given data to find the average rate of change of the items, in items per year, from $t = 0$ to $t = 7$ years. Express your answer as a decimal approximation, and interpret the meaning based on the context of the problem. Show the computations that lead to your answer.

(ii) Use the average rate of change found in (i) to estimate the number of items collected in year $t = 40$. Show the work that lead to your answer.

(C) The annual food drive continued to grow for the next 20 years, and the number of items collected each year is recorded in the table below.

***t*, years since 1980**	7	9	10	12	14	18	20
***P*(*t*), number of items collected**	160	270	310	370	440	510	580

(i) Using a graphing calculator, write a logarithmic regression model for the given data, rounding all coefficients to the nearest thousandth. Explain why a logarithmic regression model is appropriate to use in the context of this problem.
(ii) Use the logarithmic regression model found in (A) to estimate the number of items collected in year $t = 40$.
(iii) Explain why the estimation found using the average rate of change in (B)(ii) is greater than the estimation found using the logarithmic regression model in (C)(ii).

2. Let f be a function defined for all real numbers such that f is an increasing function when $x > 1$. The table gives values of $f(x)$ at selected values of x.

x	–1	0	1	2	3	4
$f(x)$	5	4	5	8	13	17

The function g is given by $g(x) = \frac{2x}{\sqrt{25 - x^2}}$.

(A) (i) The function h is defined by $h(x) = (g \circ f)(x) = g(f(x))$. Find the value of $h(1)$ as a decimal approximation, or indicate that it is not defined.

(ii) Find all values of x for which $f(x) = 5$, or indicate that there are no such values.

(B) (i) Find all values of x, as decimal approximations, for which $g(x) = 7$, or indicate that there are no such values.

(ii) Determine the end behavior of g as x increases without bound. Express your answer using the mathematical notation of a limit.

(C) (i) Determine if f has an inverse function.

(ii) Give a reason for your answer to (i) based on the definition of a function and the table of values for $f(x)$.

Part B: Graphing calculator not permitted

TIME: 30 MINUTES

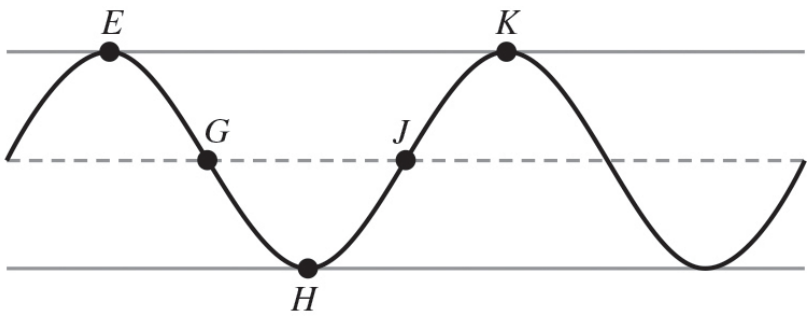
DIRECTIONS: Answer all parts of both free-response questions.

3. A spacecraft landed on a planet whose temperature fluctuates periodically. It was recorded, for the first time, that 30 hours after the landing, the minimum temperature was $4°F$. It was recorded, for the first time, that 50 hours after the landing, the maximum temperature was $120°F$.

The sinusoidal function $f(t)$ models the temperature, in $°F$, t hours after the landing.

(A) The graph of f and its dashed midline for two full cycles is shown. Five points, E, G, H, J, and K, are labeled on the graph. No scale is indicated, and no axes are presented.

Determine possible coordinates $(t, f(t))$ for the five points: E, G, H, J, and K.



(B) The function f can be written in the form $f(t) = a \sin(b(t + c)) + d$. Find values of a, b, c, and d.

(C) Refer to the graph of f in (A). The t-coordinate of E is t_0, the t-coordinate of G is t_1, the t-coordinate of H is t_2, the t-coordinate of J is t_3, and the t-coordinate of K is t_4.

(i) On the interval (t_0, t_4), state the open interval where f is positive and decreasing.

(ii) Describe how the rate of change of f is changing on the interval (t_1, t_2).

4. (A) The functions g and h are given by

$$g(x) = 2 \ln x^7 - (3 \ln x + \ln(8x^4))$$
$$h(x) = \frac{\sin 2x}{1 - \cos^2 x}$$

(i) Rewrite $g(x)$ as a single natural logarithm without negative exponents in any part of the expression. Your result should be of the form ln(expression).

(ii) Rewrite $h(x)$ as an expression in which cot x appears once and no other trigonometric functions are involved.

(B) The functions j and k are given by

$$j(x) = \tan^2 x - 1$$
$$k(x) = 5e^{(x-2)}$$

(i) Solve $j(x) = 0$ for all values of x in the interval $[0, \pi]$.

(ii) The solution to the equation $k(x) = \frac{1}{e^{-3}}$ for x can be written in the form $x = a - \ln a$.

State the value of a.

(C) The function m is given by

$$m(x) = \cos 2x + 3 \sin x$$

Find all input values in the domain of m that yield an output value of –1.

Answer Explanations

Section I

Part A

1. **(C)** Use the binomial theorem to expand $(x + 2y)^5$, where $a = x$, $b = 2y$, and $n = 5$:

$$(x+2y)^5 = {}_5C_0\,x^5(2y)^0 + {}_5C_1\,x^4(2y)^1 + {}_5C_2\,x^3(2y)^2 + {}_5C_3\,x^2(2y)^3 + {}_5C_4\,x^1(2y)^4 + {}_5C_5\,x^0(2y)^5$$

Evaluate the coefficients in front of each term either by using the formula ${}_nC_r = \frac{n!}{r!(n-r)!}$ or by copying the values from row 5 of Pascal's triangle:

$$\begin{aligned}(x+2y)^5 &= 1x^5 + 5x^4(2y) + 10x^3(4y^2) + 10x^2(8y^3) \\ &\quad + 5x(16y^4) + 1(32y^5) \\ &= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5\end{aligned}$$

The coefficient in front of $x^3\,y^2$ is 40.

2. **(C)** Given $f(1) = 4$, when $x = 1$ the function value at 1 is 4. Substitute these values in for x and $f(x)$, respectively: $1 + A + B - 3 = 4$. Given $f(-$

1) = − 6, when $x = -1$ the function value at –1 is –6. Substitute these values in for x and $f(x)$, respectively: $-1 + A - B - 3 = -6$. This is a system of two equations with two variables, A and B. Solve the system by the elimination method by adding the two equations together to eliminate B:

$$2A - 6 = -2$$
$$A = 2$$

Substitute $A = 2$ into either of the two equations, and solve for B:

$$1 + 2 + B - 3 = 4$$
$$B = 4$$

Substituting $A = 2$ and $B = 4$ gives:

$$2A + B = 2(2) + 4 = 8$$

3. **(C)** If $f(x) = x^3 - 2x - 1$, then $f(-2) = (-2)^3 - 2(-2) - 1 = -5$.

4. **(D)** The given function $f(x) = \frac{x-5}{x^2+25}$ is a rational function whose domain is all reals excluding any x-values for which the denominator is equal to 0. Setting the denominator equal to 0, $x^2 + 25 = 0$, shows that there are no real solutions, only imaginary solutions, $x = \pm 5i$. Therefore, the domain is all real numbers.

5. **(C)** The given function $g(x) = \frac{\sqrt{x-3}}{x^2-2x}$ is a rational function with a radical function. Its domain is all real numbers, excluding any x-values for which the denominator is equal to 0 or the radicand is less than 0. Set the denominator equal to 0 and solve for x:

$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0 \text{ and } x = 2$$

Therefore, the domain needs to exclude $x = 0, 2$. The radicand cannot be less than 0; otherwise the function values are imaginary. Set the radicand greater than or equal to 0:

$$x - 3 \geq 0$$
$$x \geq 3$$

Therefore, the domain is all real numbers such that $x \geq 3$.

6. **(C)** Use log laws:

$$\log\sqrt[3]{\frac{a^2b}{c}} = \log\left(\frac{a^2b}{c}\right)^{\frac{1}{3}} = \frac{1}{3}\log\left(\frac{a^2b}{c}\right) =$$
$$\frac{1}{3}(\log a^2 + \log b - \log c) = \frac{1}{3}(2\log a + \log b - \log c)$$

7. **(A)** To find the vertical asymptote(s) for the function $f(x) = \frac{x^2 - 4}{x^3 + 2x^2}$, first factor the numerator and denominator, $f(x) = \frac{(x+2)(x-2)}{x^2(x+2)}$. Remove the common factor $(x + 2)$; $x = -2$ represents a hole. The remaining factor in the denominator, x^2, indicates that there is a vertical asymptote at $x = 0$.

8. **(D)** To evaluate $f(-3)$, first identify which of the equations to use by referring to the inequalities. The input –3 is less than 0, so the first equation is used:

$$f(-3) = -\frac{1}{3}(-3)^2 + 9 = -3 + 9 = 6$$

9. **(B)** The slant asymptote of $f(x) = \frac{x^2 + 2x - 1}{x - 1}$ can be found by using long division:

$$\begin{array}{r} x + 3 \\ x - 1 \overline{)\, x^2 + 2x - 1} \\ -(x^2 - 1x) \\ \hline 3x - 1 \\ -(3x - 3) \\ \hline 2 \end{array}$$

An equivalent representation of the function is $f(x) = \frac{x^2 + 2x - 1}{x - 1} = x + 3 + \frac{2}{x - 1}$.

Therefore, the equation of the slant asymptote is $y = x + 3$.

10. **(C)** Given $\left(2, \frac{5\pi}{6}\right)$, $r = 2$, $r = 2$ and $\theta = \frac{5\pi}{6}$. This point lies in Quadrant II. Choice (A) is equivalent because even though $\theta = -\frac{\pi}{6}$ is found in Quadrant IV, since $r = -2$, the point is drawn in the reverse direction and is in Quadrant II. Choice (B) is equivalent because like Choice (A), $\theta = \frac{11\pi}{6}$ is found in Quadrant IV. Since $r = -2$, the point is drawn in the reverse direction and is in Quadrant II. Choice (C) is not equivalent because $\theta = -\frac{11\pi}{6}$ is found in Quadrant I, so Choice (C) is equivalent to $\left(2, \frac{\pi}{6}\right)$. Choice (D) is equivalent to the given point since it is found in Quadrant II.

11. **(D)** To find the zeros of $r = 2 - 4 \sin 2\theta$, set $r = 0$ and solve for θ:

$$0 = 2 - 4 \sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

Using the unit circle, $2\theta = \frac{\pi}{6}$ and $2\theta = \frac{5\pi}{6}$. Therefore, $\theta = \frac{\pi}{12}$ and $\theta = \frac{5\pi}{12}$ along with $\theta = \frac{13\pi}{12}$ and $\theta = \frac{17\pi}{12}$.

12. **(A)** Given $\cos\left[\arctan\left(-\frac{2}{3}\right)\right]$, first evaluate the inner expression. So $\arctan\left(-\frac{2}{3}\right)$ is the angle whose legs of a right triangle drawn in Quadrant II are 2 and 3. Using the Pythagorean theorem, the hypotenuse of the triangle can be found:

$$(2)^2 + (3)^2 = r^2 \rightarrow r = \sqrt{13}$$

Therefore, $\cos\theta = -\frac{3}{\sqrt{13}}$. The cosine value of this angle is negative since it is in Quadrant II. Rationalizing the denominator gives:

$$\cos\theta = -\frac{3}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

13. **(C)** Given that $\sin\theta = -\frac{1}{5}$ and $\tan\theta > 0$, since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ then $\cos\theta < 0$. Use the Pythagorean trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

Substitute $\sin\theta = -\frac{1}{5}$ and solve for $\cos\theta$:

$$\cos^2\theta + \left(-\frac{1}{5}\right)^2 = 1$$

$$\cos^2\theta = \frac{24}{25}$$

$$\cos\theta = -\frac{\sqrt{24}}{5} = -\frac{\sqrt{4}\sqrt{6}}{5} = -\frac{2\sqrt{6}}{5}$$

14. **(D)** The general equation of a sinusoidal curve is $y = a \sin b(x - c) + d$. The midline is $d = 1$. The amplitude is $a = 2$. Half of a sinusoidal curve is seen from 0 to 4, meaning a full curve is from 0 to 8. So the period is $8 = \frac{2\pi}{b}$ and $b = \frac{\pi}{4}$. A possible equation for the graph is $y = 2\sin\frac{\pi}{4}x + 1$.

15. **(D)** Using the difference identity, $\sin 5x \cos 3x - \cos 5x \sin 3x = \sin(5x - 3x) = \sin 2x$.

16. **(C)** Given $\sin x = -\frac{1}{8}$ and $\tan x < 0$, since $\tan x = \frac{\sin x}{\cos x}$ then $\cos x > 0$. To find $\cos x$, use the Pythagorean trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

Substitute $\sin\theta = -\frac{1}{8}$ and solve for $\cos\theta$:

$$\cos^2\theta + \left(-\frac{1}{8}\right)^2 = 1$$

$$\cos^2\theta = \frac{63}{64}$$

$$\cos\theta = \frac{\sqrt{63}}{8}$$

Then use the double-angle identity:

$$\sin 2x = 2\sin x \cos x = 2\left(-\frac{1}{8}\right)\left(\frac{\sqrt{63}}{8}\right)$$

$$= -\frac{2\sqrt{63}}{64} = -\frac{\sqrt{63}}{32} = -\frac{\sqrt{9}\sqrt{7}}{32} = -\frac{3\sqrt{7}}{32}$$

17. **(D)** The $= \frac{2\pi}{b}$, where $b = \frac{1}{3}$. Therefore, the $= \frac{2\pi}{\frac{1}{3}} = 6\pi$.

18. **(B)** Condense the logarithmic equation:

$$\log_a 3 - \log_a b = c$$

$$\log_a \frac{3}{b} = c$$

Rewrite in exponential form:

$$a^c = \frac{3}{b}$$

Solve for b:

$$b = \frac{3}{a^c}$$

19. **(C)** To determine if a function is even or odd, substitute $-x$ for x:

$$\begin{aligned} g(-x) &= (-x)^7 + (-x)^3 + \sin(-x) \\ &= -x^7 - x^3 - \sin x = -g(x) \end{aligned}$$

Since $g(-x) = -g(x)$, the function is odd and is symmetric with respect to the origin.

20. **(A)** To write the equation of the line, determine the slope by calculating the average rate of change:

$$\frac{\Delta y}{\Delta x} = \frac{-11 - (-7)}{5 - (-3)} = \frac{-4}{8} = -\frac{1}{2} = m$$

Substitute either input-output pair and the slope into the point-slope formula:

$$y - y_1 = m(x - x_1)$$

Using the first condition gives the equation:

$$y - (-7) = -\frac{1}{2}(x - (-3))$$

$$y + 7 = -\frac{1}{2}(x + 3)$$

21. **(B)** Determine the composition:

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 4) = \sqrt{x^2 - 4} + 4$$

22. **(D)** To determine $\lim_{x \to -\infty} g(x)$, find the end behavior of the function on the left. Since the leading coefficient is negative and the degree is the sum of the multiplicities, 5, which is odd, the function rises on the left and falls on the right. In other words, $\lim_{x \to -\infty} g(x) = +\infty$.

23. **(B)** The graph of $g(x) = -5^{-x} - 2$ is the transformation of the parent function $f(x) = 5^x$. The graph of $f(x)$ is reflected over the x- and y-axes and shifted down 2 units. After the transformation, $\lim_{x \to \infty} f(x) = -2$.

24. **(A)** A reflection over the y-axis transforms the equation of a function by replacing x with $-x$. Therefore, the transformed equation is $g(x) = 1 - 3^{-x}$.

25. **(C)** For a function to have an inverse function, it must be one-to-one. The function in Choice (C) is not one-to-one since it fails the horizontal line test.

26. **(D)** The given sequence is arithmetic since there is a common difference, $d = 19$. Use the general arithmetic sequence formula:

$$\begin{aligned} a_n &= a_1 + d(n-1) = 19 + 19(n-1) \\ &= 19 + 19n - 19 = 19n \end{aligned}$$

27. **(A)** The graphs of polynomial functions pass through the x-axis at x-intercepts whose factors have odd multiplicity. The only factor with an odd multiplicity is $x = 5$. Therefore, the graph of the polynomial function passes through the x-intercept at (5, 0) and is tangent to the x-axis at (–5, 0).

28. **(C)** To evaluate infinite limits of rational functions, determine their end behavior by considering the degree of both the numerator and the denominator of the expression. The degree of the numerator and denominator are both 2. Since they are equal, the limit is the ratio of their leading coefficients, $\frac{3}{5}$.

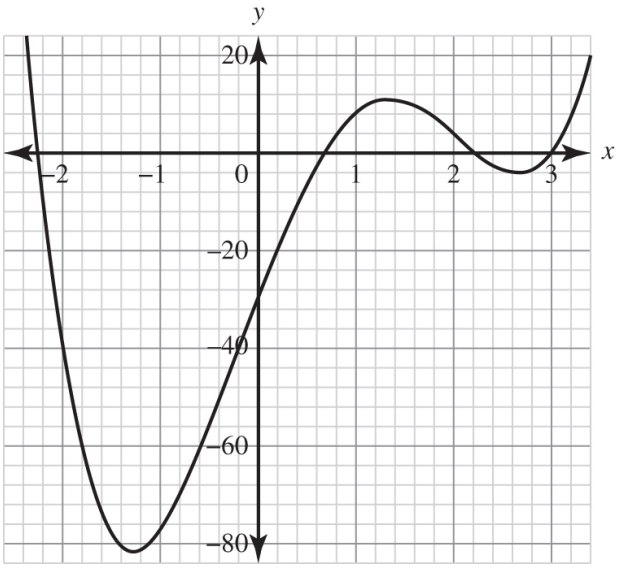
Part B

29. **(D)** Graphically, the greatest average rate of change is equal to the steepest slope of a secant line. The lines can be drawn in to see the steepest slope, or estimations of the coordinates can be used to calculate the slope of the secant for each choice as shown in the table below.

Points	Slope of Secant Line
$a(-3, 0)$ and $c(-1, -3.5)$	$m = \frac{-3.5 - 0}{-1 - (-3)} = \frac{-3.5}{2} = -1.75$
$b(-2, -3.5)$ and $c(-1, -3.5)$	$m = \frac{-3.5 - (-3.5)}{-1 - (-2)} = \frac{0}{1} = 0$
$c(-1, -3.5)$ and $d(1.75, 0)$	$m = \frac{0 - (-3.5)}{1.75 - (-1)} = \frac{3.5}{2.75} \approx 1.2727$
$f(4, -1)$ and $h(5.5, 2)$	$m = \frac{2 - (-1)}{5.5 - 4} = \frac{3}{1.5} = 2$

The sign indicates the direction of the secant line. So the slope that has the greatest absolute value is between f and h.

30. **(D)** The absolute maximum value of a function is the highest y-value in the range of a given function and can occur at relative maxima or at endpoints of the interval. Below is the graph of $f(x)$.



From the graph it is hard to tell if the absolute maximum value occurs at Choice (A), (C), or (D). In this case, finding the y-values for each of the x-values is a way to choose the correct answer. The table of values is given below.

x	$f(x)$
−2.28	9.2604
−1.27	−84.03
1.35	10.748
3.25	10.777

The highest y-value is 10.777, which occurs at the endpoint of the interval and would be the x-value with the absolute maximum value for $f(x)$.

31. **(A)** Using log laws:

$$\begin{aligned}\log_a 24 &= \log_a(3 \cdot 2 \cdot 2 \cdot 2)\\ &= \log_a 3 + \log_a 2 + \log_a 2 + \log_a 2\\ &= \log_a 3 + 3\log_a 2\\ &= 0.6826 + 3(0.4307) = 1.9747\end{aligned}$$

32. **(A)** To solve for x given the equation $3^2x = 5^{x-1}$, use a graphing calculator. Let Y1 = 3^{2x} and Y2 = 5^{x-1}. To find their point of intersection, press 2ND TRACE and select 5:intersect. The point of intersection is (−2.738153, 0.00243869).

33. **(B)** When graphed, the complex number $3 - 8i$ is in Quadrant IV. To find r, use the Pythagorean theorem:

$$(3)^2 + (8)^2 = r^2$$

$$r = \sqrt{73}$$

This eliminates Choices (C) and (D). To find the angle, use the calculator to find the reference angle:

$$\tan^{-1}\frac{8}{3} \approx 1.2120256 \text{ radians}$$

Since the angle is in Quadrant IV, $\theta \approx 2\pi - 1.2120256 \approx 5.071159$ radians. The angle in Choice (B), $\frac{97\pi}{60} \approx 5.071159$ radians, is correct.

34. **(B)** Use the Pythagorean trigonometric identity $\cos^2 x + \sin^2 x = 1$ to substitute in for $\cos^2 x$. Then the given equation $6\cos^2 x - 5\sin x - 2 = 0$ becomes $6(1 - \sin^2 x) - 5\sin x - 2 = 0$. Write the equation in standard form:

$$6\sin^2 x + 5\sin x - 4 = 0$$

Factor the trinomial:

$$(3\sin x + 4)(2\sin x - 1) = 0$$

Set each factor equal to 0 and solve for $\sin x$:

$$\sin x = -\frac{4}{3} \text{ and } \sin x = \frac{1}{2}$$

There is no value of x for which the first equation can be true since the range of $\sin x$ is $[-1, 1]$.

Therefore, the only solution comes from $\sin x = \frac{1}{2}$. Using the unit circle, $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

35. **(A)** $\cos(\sin^{-1} x)$ means that $\theta = \sin^{-1} x$, so $\sin\theta = x$. Using the Pythagorean identity, solve for $\cos\theta$:

$$x^2 + \cos^2\theta = 1$$

Therefore:

$$\cos\theta = \sqrt{1 - x^2}$$

36. **(B)** When an investment is compounded continuously, it is modeled by the equation $f(t) = a \cdot e^{rt}$. In this case, $a = 1{,}000$, $f(t) = 2{,}000$, and $r = 0.058$. Substitute the values:

$$2{,}000 = 1{,}000\, e^{(0.058)t}$$

To solve for t, you can use technology to graph the left and right sides of the equation and find their point of intersection. Algebraically, isolate the exponential expression and use natural logs to solve for t:

$$2 = e^{0.058t}$$

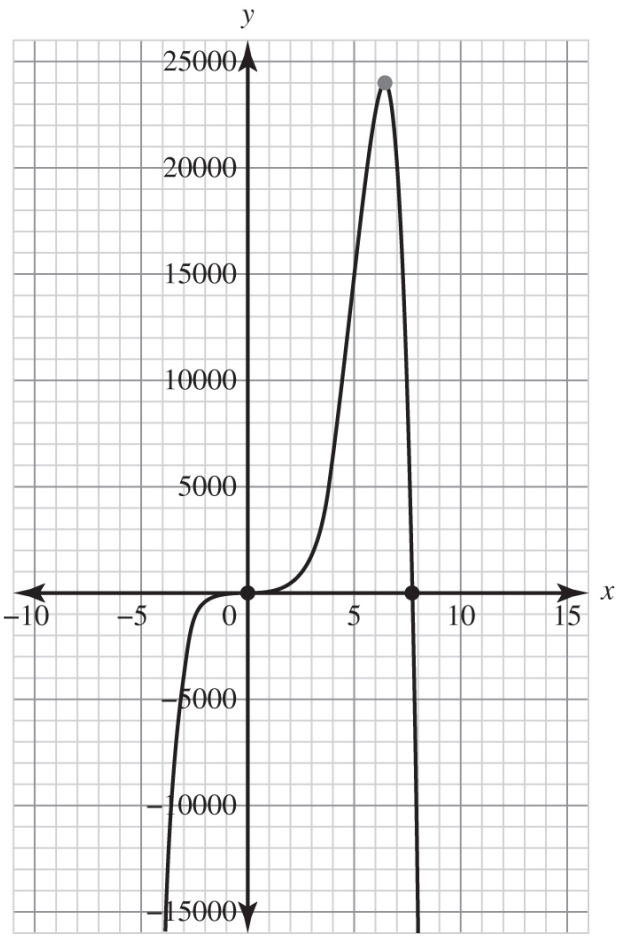
$$t = \frac{\ln 2}{0.058} \approx 11.9508$$

37. **(D)** Given the function $h(t) = -16t^2 + 50t$, set the equation equal to 28 and solve for t. This can be done graphically by finding the point of intersection using technology. There are two places where the object reaches a height of 28 feet. The second point is chosen as this is also where the object is traveling downward, (2.394, 28).

38. **(C)** This represents a geometric sequence with the equation $g\ n = 15(0.85)^n$. To find the height after the third bounce, substitute $n = 3$ into the equation:

$$g_3 = 15(0.85)^{(3)} = 9.2$$

39. **(B)** Using technology, graph the function and look at where the graph changes concavity.



The graph changes from concave down to concave up at $x = 0$ and then changes from concave up to concave down at $x = 5$.

40. **(C)** For a residual plot, there should be no observable pattern and a similar distribution of residuals above and below the x-axis.

Section II

Part A

1. (A)(i) Substitute the coordinate pairs (0, 50) and (7, 160) into the equation $F(t) = a + b \ln(t + 1)$:

$$50 = a + b \ln(0 + 1)$$
$$160 = a + b \ln(7 + 1)$$

(ii) To solve for *a* and *b*, simplify the first equation by substituting 0 for ln(1):

$$50 = a + b(0)$$
$$50 = a$$

Substitute 50 for *a* into the second equation to solve for *b*:

$$160 = 50 + b\ln(7 + 1)$$
$$160 = 50 + b\ln(8)$$

Solve for *b* by subtracting 50 and dividing both sides by ln(8):

$$110 = b\ln(8)$$
$$b = \frac{110}{\ln(8)} \approx 52.899$$

(B)(i) Average rate of $= \frac{160 - 50}{7 - 0} = \frac{110}{7} \approx 15.714$

On average, there is an increase of 15.714 items collected per year.

(ii) To find the approximate number of items collected when $t = 40$, write a linear approximation equation using the given ordered pair (0, 50) and the average rate of change, 15.714:

$$y - 50 = 15.714(x - 0)$$

Substitute 40 for *x* and solve for *y* :

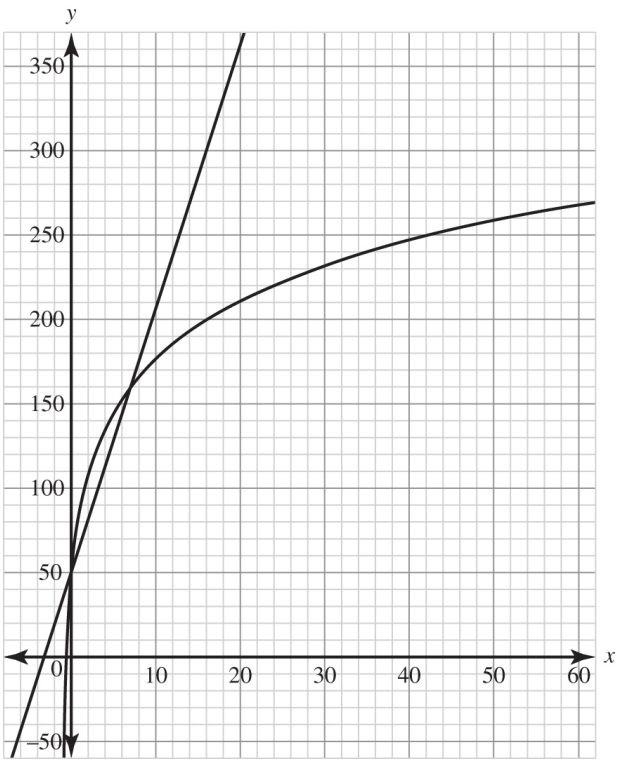
$$y - 50 = (15.714)(40 - 0) \rightarrow y = 678.56$$

The number of items collected in year $t = 40$ is approximately 679 items.

(C) (i) Using a graphing calculator, press STAT and ENTER to edit L1 and L2 by entering the *t* and *P*(*t*) values. After entering the values, press STAT and then CALC and 9:LnReg. Select Calculate. The equation is $y = –577.321 + 382.753 \ln(x)$. A logarithmic model is appropriate since the values are increasing as the independent variable increases, and the r^2 value is .99466, which is close to 1.

(ii) The logarithmic regression model is $y = 50 + 52.899 \ln(t + 1)$. Substituting in $t = 40$, $y = 50 + 52.899 \ln(40 + 1) \approx 246.444$. The number of items collected in year $t = 40$ is approximately 246 items.

(iii) Since $t = 40$ is outside the given interval, $0 < t < 7$, approximations found using the secant line (linear approximation model) will be greater than those found using the logarithmic model. This is because the logarithmic model increases at a decreasing rate since it is concave down, while the secant line has a constant rate of increase. This can be seen by looking the graph of the logarithmic model and the secant line shown below.



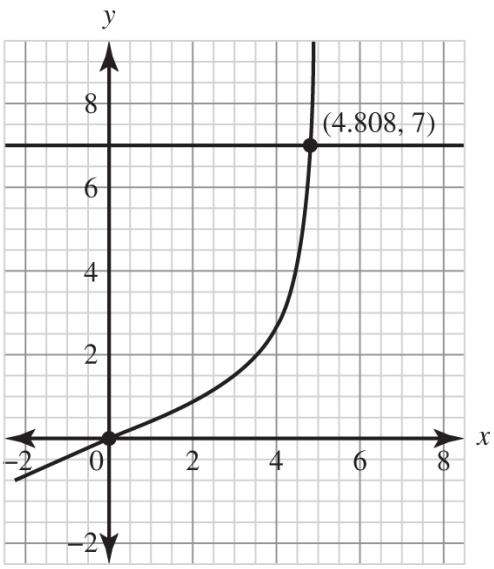
2. (A)(i) $h(1) = (g \circ f)(1) = g(f(1)) = g(5) = \frac{2(5)}{\sqrt{25 - (5)^2}} = \frac{10}{\sqrt{25 - 25}}$, which is undefined.

(ii) The values of x for which $f(x) = 5$ are the x-values that correspond to the y-values that are 5. The x-values are –1 and 1.

(B)(i) Set the g(x) equation equal to 7:

$$7 = \frac{2x}{\sqrt{25 - x^2}}$$

Using a graphing calculator, graph both sides of the equation and find the point of intersection:



When $g(x) = 7$, $x \approx 4.808$.

(ii) As seen on the graph of $g(x)$, as x increases without bound, so does $g(x)$. Therefore:

$$\lim_{x \to \infty} g(x) = \infty$$

(C)(i) f does not have an inverse function.

(ii) Using the table of values for $f(x)$, f does not have an inverse function since there are two repeating y-values. For a function to have an inverse function, every x-value can have one and only one y-value and every y-value can have one and only one x-value.

Part B

3. (A) It was given that 30 hours after landing, the minimum temperature was 4°F. So, point H can have coordinates (30, 4). It was also given that 50 hours after landing, the maximum temperature was 120°F. So, point

K can have coordinates (50, 120). Midway between the minimum and maximum values is $\frac{4+120}{2} = \frac{124}{2} = 62$. This represents the *y*-coordinates of points *G* and *J*. To find the *x*-coordinates, take the difference between 50 and 30 and divide by 2: $\frac{50-30}{2} = \frac{20}{2} = 10$. Therefore, point *J* can have coordinates (40, 62), point *G* can have coordinates (20, 62), and point *E* can have coordinates (10, 120). (Note that *t*-coordinates may vary.)

(B) For function *f*:

- *a* represents the amplitude or distance from the midline to the maximum value: $a = \frac{120-4}{2} = \frac{116}{2} = 58$
- *b* represents the number of full cycles from 0 to 2π: $b = \frac{2\pi}{50-10} = \frac{2\pi}{40} = \frac{\pi}{20}$
- *c* represents the horizontal shift from the origin: $c = 0$
- *d* represents the vertical shift up from the origin: $d = 62$

Therefore, $f(t) = 58\sin\left(\frac{\pi}{20}(t+0)\right) + 62$.

(C)(i) *f* is positive and decreasing on the interval (t_0, t_2).

(ii) Since the graph of *f* is concave up on the interval (t_1, t_2), the rate of change of *f* is increasing on the interval (t_1, t_2).

4. (A)(i) Use log laws to condense:

$$g(x) = 2\ln x^7 - (3\ln x + \ln(8x^4))$$
$$g(x) = \ln(x^{7\cdot 2}) - \ln(x^3) - \ln(8x^4)$$
$$g(x) = \ln\frac{x^{14}}{x^3 \cdot 8x^4}$$
$$g(x) = \ln\frac{x^{14}}{8x^7}$$
$$g(x) = \ln\frac{x^7}{8}$$

(ii) Use the double-angle identity and the Pythagorean identity:

$$h(x) = \frac{\sin 2x}{1 - \cos^2 x}$$
$$h(x) = \frac{2 \sin x \cos x}{\sin^2 x}$$
$$h(x) = \frac{2 \cos x}{\sin x}$$
$$h(x) = 2 \cot x$$

(B)(i) First set the function equal to 0:

$$\tan^2 x - 1 = 0$$

Factor by the difference of perfect squares:

$$(\tan x - 1)(\tan x + 1) = 0$$

Set each factor equal to 0:

$$\tan x - 1 = 0 \qquad \tan x + 1 = 0$$

Solve for tan *x*:

$$\tan x = 1 \qquad \tan x = -1$$

To solve for *x*, evaluate the inverse tangent of each equation:

$$x = \tan^{-1}(1) \qquad x = \tan^{-1}(-1)$$

Using the unit circle, for $x = \tan^{-1}(1)$, $x = \frac{\pi}{4}$ in the interval $[0, \pi]$.

Using the unit circle, for $x = \tan^{-1}(-1)$, $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ in the interval $[0, \pi]$.

(ii) Substitute $5\,e^{(x-2)}$ in for $k(x)$ into the equation $k(x) = \frac{1}{e^{-3}}$:

$$5\,e^{(x-2)} = \frac{1}{e^{-3}}$$

Use the law of exponents on the right side of the equation:

$$5\,e^{(x-2)} = e^{3}$$

Divide both sides of the equation by 5:

$$e^{(x-2)} = \frac{e^3}{5}$$

Take the natural logarithm of both sides of the equation:

$$\ln e^{(x-2)} = \ln \frac{e^3}{5}$$

Simplify using log laws:

$$x - 2 = \ln e^3 - \ln 5$$
$$x - 2 = 3 - \ln 5$$

Isolate *x* by adding 2 to both sides of the equation:

$$x = 5 - \ln 5$$
$$a = 5$$

(C) Set the equation equal to –1:

$$\cos 2x + 3 \sin x = -1$$

Use the double-angle identity:

$$1 - 2\sin^2 x + 3 \sin x = -1$$

Write the quadratic in standard form set equal to 0:

$$2\sin^2 x - 3 \sin x - 2 = 0$$

Factor the quadratic equation:

$$2\sin^2 x - 4 \sin x + 1 \sin x - 2 = 0$$
$$2 \sin x(\sin x - 2) + 1(\sin x - 2) = 0$$
$$(2 \sin x + 1)(\sin x - 2) = 0$$

Set each factor equal to 0:

$$2 \sin x + 1 = 0 \qquad \sin x - 2 = 0$$

Solve for sin *x*:

$$\sin x = -\frac{1}{2} \qquad \sin x = 2$$

To solve for *x*, evaluate the inverse sine of each equation:

$$x = \sin^{-1}\left(-\frac{1}{2}\right) \qquad x = \sin^{-1}(2)$$

Using the unit circle, for $x = \sin^{-1}\left(-\frac{1}{2}\right)$, the reference angle is $x_{reference} = \frac{\pi}{6}$. Therefore, $x = \pi + \frac{\pi}{6} \rightarrow x = \frac{7\pi}{6} + 2k\pi$ or $x = 2\pi - \frac{\pi}{6} \rightarrow x = \frac{11\pi}{6} + 2k\pi$, where *k* is any integer.

Using the unit circle, for $x = \sin^{-1}(2)$, *x* is undefined since the domain of $\sin^{-1} x$ is [–1, 1].

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Practice Test 2

Section I

Part A: Graphing calculator not permitted

TIME: 80 MINUTES

DIRECTIONS: For each question or incomplete statement, select the choice that best answers the question or completes the statement.

1. What is the fifth term in the expansion of $(3a - b)^6$?

   (A) $135a^2b^4$
   (B) $540a^3\ b^3$
   (C) $-18ab^5$
   (D) $-135a^2b^4$

2. Simplify: $\ln \sqrt[3]{e^2x}$

   (A) $\frac{2e}{3} + \frac{1}{3}\ln x$
   (B) $\frac{2}{3} + \ln\frac{x}{3}$
   (C) $\frac{2}{3} + \frac{1}{3}\ln x$

(D) $\frac{2e}{3} + \ln\frac{x}{3}$

3. If $\csc x \neq -1$, which of the following is equivalent to $\frac{\cot^2 x}{1 + \csc x}$?

(A) $1 - \csc x$
(B) $\csc x - 1$
(C) $-\csc x$
(D) $\csc x - \cot x$

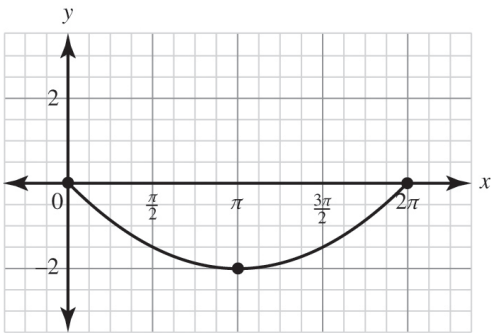
4. If $\sin\theta = b$, then what is $\sin\theta \cdot \cos\theta \cdot \tan\theta$?

(A) 1
(B) $\frac{1}{b}$
(C) b
(D) b^2

5. If $f(\theta) = a\cos\theta + b$, what is the minimum value of $f(\theta)$?

(A) $b - a$
(B) $\frac{b-a}{2}$
(C) $b + a$
(B) $\frac{b+a}{2}$

6. Which of the following equations could represent the graph shown below?



(A) $y = 2\sin(2x)$
(B) $y = -2\sin\left(\frac{1}{2}x\right)$

(C) $y = -\frac{1}{2}\sin(2x)$

(D) $y = \frac{1}{2}\sin(2x)$

7. Which of the following expressions is equivalent to $\csc\left(-\frac{\pi}{2}\right)$?

(A) $2\cos\frac{\pi}{3}$

(B) $\tan \pi$

(C) $2\sin\frac{5\pi}{6}$

(D) $\cos \pi$

8. In which quadrant is the terminal side of angle θ located if the graphs of $y = \sin \theta$ and $y = \cos \theta$ are both decreasing when angle θ is increasing?

(A) I

(B) II

(C) III

(D) IV

9. Evaluate: $\csc\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

(A) $\frac{5}{3}$

(B) $\frac{5}{4}$

(C) $\frac{4}{5}$

(D) $\frac{3}{5}$

10. For the expression $k - \frac{1}{\sec^2 x} = \sin^2 x$ to be an identity, what does k equal?

(A) 1

(B) 0

(C) $\cos^2 x$

(D) $\csc^2 x$

11. What is the expression $\frac{\sin 2\theta}{2\tan\theta}$ equivalent to?

(A) $\cot^2 \theta$

(B) $\sin^2 \theta$

(C) $\cos^2 \theta$

(D) $\sec^2 \theta$

12. In the interval $0 \le x < 2\pi$, what are the solutions of the equation $\sin^2 x = \sin x$?

(A) $0, \frac{\pi}{2}, \pi$

(B) $\frac{\pi}{2}, \frac{3\pi}{2}$

(C) $0, \frac{\pi}{2}, \frac{3\pi}{2}$

(D) $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$

13. What is the expression $\frac{\sin\left(x + \frac{\pi}{2}\right)}{-\sin x}$ equivalent to?

(A) -1

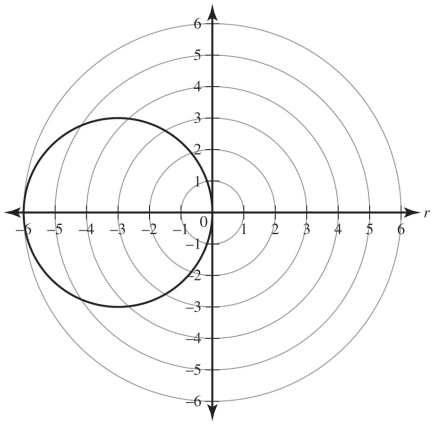
(B) 1

(C) $-\cot x$

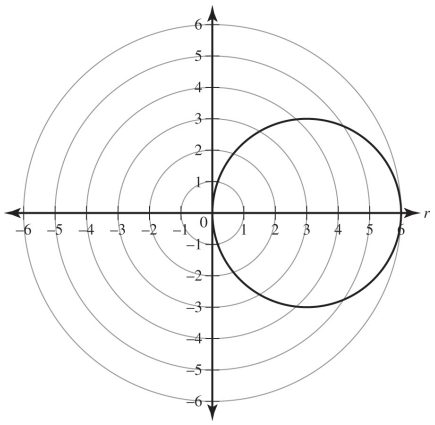
(D) $\cot x$

14. Which of the following choices represents the graph of $r = 6 \cos \theta$?

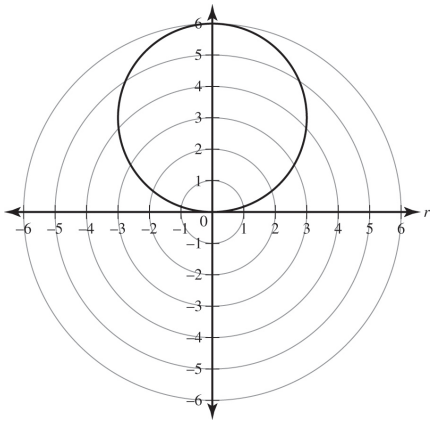
(A)



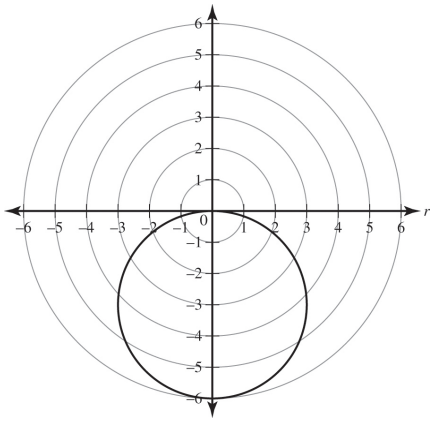
(B)



(C)



(D)



15. Which of the following points does *not* change the location of the point $\left(4, \frac{11\pi}{18}\right)$ in polar coordinates?

(A) $\left(4, \frac{29\pi}{18}\right)$

(B) $\left(-4, \frac{10\pi}{9}\right)$

(C) $\left(-4, \frac{47\pi}{18}\right)$

(D) $\left(4, \frac{47\pi}{18}\right)$

16. Given the polar coordinates $\left(3, -\frac{3\pi}{4}\right)$, find the rectangular coordinates of this point.

(A) $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

(B) $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$

(C) $\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

(D) $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$

17. If $f(x) = x^3 - 3x^2 - 2x + 5$ and $g(x) = 2$, then what does $g(f(x))$ equal?

(A) $2x^3 - 6x^2 - 2x + 10$

(B) $2x^2 - 6x + 1$

(C) -3

(D) 2

18. Which of the following functions is *not* odd?

(A) $f(x) = \sin(x)$

(B) $g(x) = \sin(2x)$

(C) $h(x) = x^5 + 1$

(D) $j(x) = \frac{x}{x^2 + 3}$

19. At what value(s) of x do the graphs of $y = x + 2$ and $y^2 = 4x$ intersect?

(A) -2 and 2

(B) -2

(C) 2

(D) none of these

20. If $\log_b(3^b) = \frac{b}{2}$, then what does b equal?

(A) $\frac{1}{9}$

(B) $\frac{1}{3}$

(C) 3

(D) 9

21. If $f^{-1}(x)$ is the inverse of $f(x) = 2e^{-x}$, then what does $f^{-1}(x)$ equal?

(A) $\ln\left(\frac{2}{x}\right)$

(B) $\ln\left(\frac{x}{2}\right)$

(C) $\frac{1}{2}\ln(x)$

(D) $\ln(2 - x)$

22. Find $f(x + h)$ when $f(x) = 2x^2 + 4x - 1$.

(A) $2x^2 + 2h^2 + 8x + 8h - 1$

(B) $2x^2 + 2h^2 + 4x + 4h - 1$

(C) $2x^2 + 2xh + 2h^2 + 4x + 4h - 1$

(D) $2x^2 + 4xh + 2h^2 + 4x + 4h - 1$

23. For $f(x) = \begin{cases} 8x + 1, x < 1 \\ 4x, 4 \le x \le 6, \\ 4 - 2x, x > 6 \end{cases}$, evaluate $f(6)$.

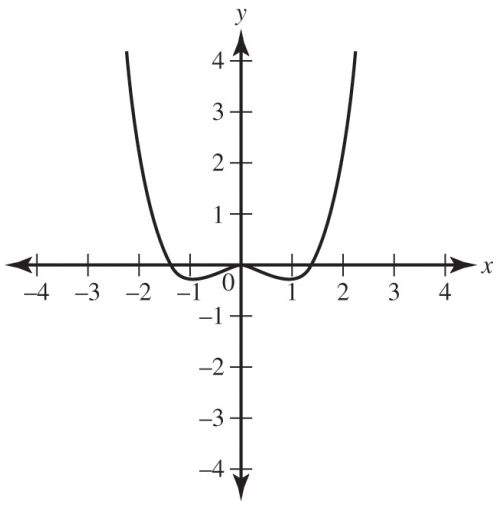
(A) -8

(B) 6

(C) 24

(D) 49

24. Use the graph of $h(x)$ below to determine the intervals where $h(x)$ is increasing and where $h(x)$ is decreasing.



(A) Increasing: (−1, 1); decreasing: (−∞, −1) ∪ (1, ∞)

(D) Increasing: (−1, 0) ∪ (1, ∞); decreasing: (−∞, −1) ∪ (0, 1)

(D) Increasing: (0, ∞); decreasing: (−∞, 0)

(D) Increasing: (−1, ∞); decreasing: (−∞, −1)

25. Using the tables below, evaluate $(f \circ g)(4)$.

x	13	9	5	7
$f(x)$	26	18	10	14

x	6	4	7	5
$g(x)$	11	7	13	9

(A) 4

(D) 7

(D) 14

(D) 18

26. Find the horizontal asymptote, if any, for the rational function $g(x) = \frac{7x^2 - 3x - 6}{6 - 8x + 3x^2}$.

(A) $y = 0$

(B) $y = \frac{7}{3}$

(C) $y = \frac{3}{8}$

(D) no horizontal asymptote

27. A farmer has 1,400 yards of fencing to enclose a rectangular garden. Express the area, A, of the rectangle as a function of the width, x, of the rectangle. What is the domain of A?

(A) $A(x) = -x^2 + 700x; 0 < x < 700$

(B) $A(x) = -x^2 + 700x; 0 < x < 1{,}400$

(C) $A(x) = x^2 + 700x; 0 < x < 700$

(D) $A(x) = -x^2 + 1{,}400x; 0 < x < 1{,}400$

28. If $x = -2$ is a real zero of the polynomial $f(x) = x^3 + 12x^2 + 46x + 52$, write $f(x)$ as a product of linear factors.

(A) $f(x) = (x + 1)(x + 5 + i\sqrt{3})(x - 2 - i\sqrt{3})$

(B) $f(x) = (x - 1)(x + 5 + i\sqrt{3})(x + 5 - i\sqrt{3})$

(C) $f(x) = (x + 2)(x + 5 + i)(x + 5 - i)$

(D) $f(x) = (x + 2)(x + 5 + i)(x - 5 - i)$

Part B: Graphing calculator required

TIME: 40 MINUTES

DIRECTIONS: For each question or incomplete statement, select the choice that best answers the question or completes the statement.

29. A deposit of $12,000 is made in an account that earns 7.6% interest compounded quarterly. The balance in the account after n quarters is given by the sequence $a_n = 12{,}000\left(1 + \frac{0.076}{4}\right)^n, n = 1, 2, 3, \ldots$ Find the balance in the account after 5 years.

(A) $6,051.40

(B) $7,285.40

(C) $13,184.15

(D) $17,484.97

30. If $\log_{0}.3(x - 1) < \log_{0}.09(x - 1)$, then x lies in which of the following intervals?

(A) $(2, \infty)$
(B) $(-2, -1)$
(C) $(-\infty, 2)$
(D) $(1, 2)$

31. If θ is an angle in standard position and its terminal side passes through the point $P(-0.8, 0.6)$ on the unit circle, which of the following is a possible radian value of θ to the nearest hundredth?

(A) 2.22
(B) 2.50
(C) 5.36
(D) 5.64

32. The owner of a video game store has determined that the cost, C, in dollars, of operating the store is approximately modeled by the function $C(x) = 2x^2 - 22x + 760$, where x is the number of video games rented daily. Find the lowest cost to the nearest dollar.

(A) $518
(B) $639
(C) $700
(D) $821

33. The height, h, in meters of a rock thrown straight up at t seconds is described by the equation $h(t) = 2 + 20t - 4.9t^2$. Find the average rate of change of the height of the rock over the first 2 seconds of flight.

(A) -10.2
(B) -0.0980
(C) 0.0980
(B) 10.2

34. Find the solution set for the rational inequality $\frac{x+2}{2x-1} > 5$.

(A) $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{7}{9}, \infty\right)$

(B) $(-\infty, 2)$

(C) $\left(\frac{1}{2}, 2\right)$

(D) $\left(\frac{1}{2}, \frac{7}{9}\right)$

35. The table below shows selected values of (x, y) for a continuous polynomial relation. Based on the table, within which interval of x must there exist a zero of the polynomial?

x	–3	–1	1	3	5	7
y	4.24	-10.18	-16.6	-15.02	-5.44	6.35

(A) $(-1, 1)$

(D) $(1, 3)$

(D) $(3, 5)$

(D) $(5, 7)$

36. Find the domain of the function $f(x) = \sqrt{x-a} - \frac{1}{\sqrt{b-x}}, 0 < a < b$.

(A) $[a, b)$

(B) $(a, b]$

(C) (a, b)

(D) $(-\infty, \infty)$

37. Find the difference of the largest and smallest solutions of the equation $5x^4 + x^2 - 1 = 2^x$ rounded to 2 decimal places.

(A) 0

(B) 1.13

(C) 1.34

(D) 1.50

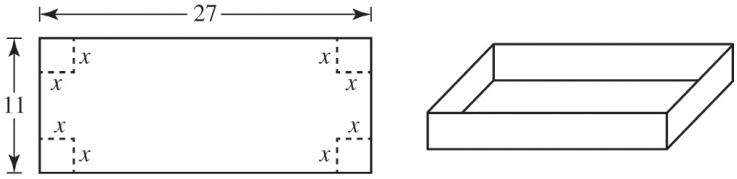
38. The population of Old Bethpage is given in the table below.

Year	1980	1990	2000	2010
Population (in thousands)	2	4	8	17

If the same pattern of population growth continued, what was the population of Old Bethpage in the year 2024, to the nearest thousand?

(A) 20,000
(D) 32,000
(D) 45,000
(D) 64,000

39. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 11 inches by 27 inches by cutting out equal squares of side x at each corner and then folding up the sides as seen in the figure below.



What would be the length of the side of the square for the volume of the box to be an absolute maximum?

(A) 2.41
(D) 5.5
(D) 5.8
(D) 330.35

40. Given the rectangular coordinates $(4, -4\sqrt{3})$, find the polar coordinates of this point.

(A) $\left(4, \frac{5\pi}{3}\right)$

(B) $\left(4, \frac{11\pi}{6}\right)$

(C) $\left(8, \frac{11\pi}{6}\right)$

(D) $\left(8, \frac{5\pi}{3}\right)$

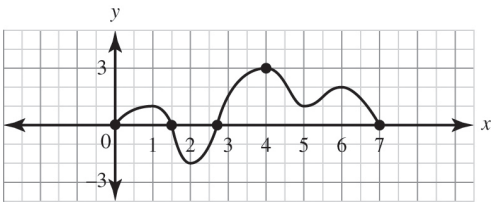
Section II

Part A: Graphing calculator required

TIME: 30 MINUTES

DIRECTIONS: Answer all parts of both free-response questions.

1. Below is the graph of the function $y = f(x)$ on its domain $0 \le x \le 7$. The points (0, 0), (1.5, 0), (2.7, 0), and (7, 0) are on the graph of f.



Graph of f

The function g is given by $g(x) = 5.212\ (6)^{-x}$.

(A) (i) The function h is defined by $h(x) = (g \circ f)(x) = g(f(x))$. Find the value of $h(4)$ as a decimal approximation, or indicate that it is not defined.

(ii) Find all zeros for $f(x)$, or indicate that there are no such values.

(B) (i) Find all values of x, as decimal approximations, for which $g(x) = -2$, or indicate that there are no such values.

(ii) Determine the end behavior of g as x increases without bound. Express your answer using the mathematical notation of a limit.

(C) (i) Determine if f has an inverse function.

(ii) Give a reason for your answer to (i) based on the definition of a function and the graph of $y = f(x)$.

2. As a result of advances in medical science, life expectancy has been increasing in most developed countries since the beginning of the 20th century. In the year 1900 ($t = 0$), the average life expectancy was 47.3 years. In the year 2010 ($t = 110$), the average life expectancy was 78.7 years.

The average life expectancy in a given year can be modeled by the function L, given by $L(t) = a + b \ln (t + 1)$, where $L(t)$ is the average life expectancy, in years, t years from the year 1900.

(A) (i) Use the given data to write two equations that can be used to find the values for constants a and b in the expression for $L(t)$.

(ii) Find the values for a and b as decimal approximations.

(B) (i) Use the given data to find the average rate of change of the average life expectancy, in years per year, from $t = 0$ to $t = 110$. Express your answer as a decimal approximation. Show the computations that lead to your answer.

(ii) Interpret the meaning of your answer from (i) based on the context of the problem.

(iii) Use the average rate of change found in (i) to estimate what the average life expectancy was in the year 1950. Show the work that lead to your answer.

(C) (i) Use the model found in (A)(ii) to estimate the average life expectancy in the year 1950.

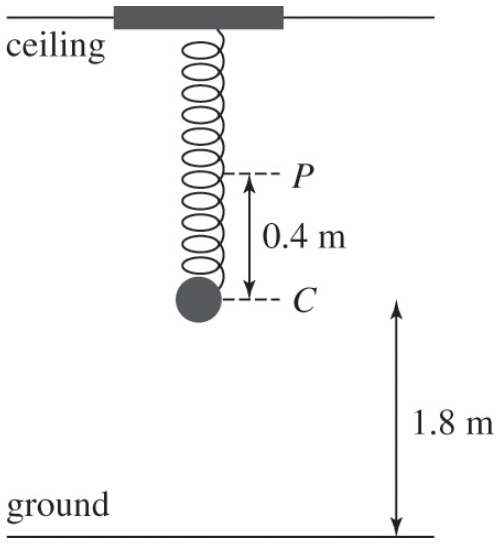
(ii) Compare the values found in (B)(iii) and (C)(i), and explain why their relationship occurs for $0 < t < 110$.

Part B: Graphing calculator not permitted

TIME: 30 MINUTES

DIRECTIONS: Answer all parts of both free-response questions.

3. As shown in the following diagram, a ball is attached to the end of a spring that is suspended from a ceiling.

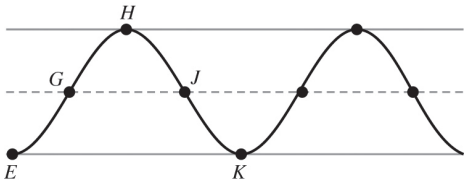


Point C in the center of the ball is 1.8 meters from the ground. When the ball is released, it rises to point P, which is 0.4 meters from point C. It takes the ball $\frac{3}{2}$ seconds to go from its resting position at point C, to point P, and back to point C. After the ball is released from rest, the height of point C above the ground periodically increases and decreases.

The sinusoidal function $h(t)$ models the height of point C above the ground, in meters, as a function of time, t, in seconds.

(A) The graph of h and its dashed midline for two full cycles is shown. Five points, E, G, H, J, and K, are labeled on the graph. No scale is indicated, and no axes are presented.

Determine possible coordinates $(t, h(t))$ for the five points: E, G, H, J, and K.



(B) The function h can be written in the form $h(t) = a \sin(b(t + c)) + d$. Find the values of constants a, b, c, and d.

(C) Refer to the graph of h in (A). The t-coordinate of E is t_1, the t-coordinate of G is t_2, the t-coordinate of H is t_3, the t-coordinate of J is t_4, and the t-coordinate of K is t_5.

(i) On which interval ($t\ a$, $t\ b$) is h positive and decreasing?

(ii) Describe how the rate of change of h is changing on the interval (t_3, t_4).

4. (A) The functions f and g are given by

$$f(x) = \log^{16}(x^2 - x + 4) - \log_{16}(2x + 11)$$

$$g(x) = -10\ e\ 2-2x - 6$$

(i) Rewrite $f(x)$ as a single logarithm base 16 without negative exponents in any part of the expression. Your result should be of the form $\log^{16}$(expression).

(ii) Using your answer from (i), solve $f(x) = \frac{3}{4}$ for all values of x in the domain of f.

(iii) Solve $g(x) = -66$ for all values of x in the domain of g.

(B) The function k is given by

$$k(x) = \cos(2x) - \sin(x)$$

(i) Rewrite $k(x)$ as an expression in terms of $\sin(x)$ and in which no other trigonometric functions are involved.

(ii) Using your answer from (i), solve $k(x) = 0$ for all values of x in the interval $[-\pi, \pi]$.

(C) The function h is given by

$$h(x) = \tan\left(\cos^{-1}\left(\frac{\sqrt{3}}{x}\right)\right)$$

Find all values in the domain of h that yield an output value of $\frac{\sqrt{3}}{3}$.

Answer Explanations

Section I

Part A

1. **(A)** Use the binomial theorem to expand $(3a - b)^6$, where $a = 3a$, $b = -b$, and $n = 6$:

$$\begin{aligned}(3a - b)^6 = {}&{}_6C_0(3a)^6(-b)^0 + {}_6C_1(3a)^5(-b)^1 + \\ &{}_6C_2(3a)^4(-b)^2 + {}_6C_3(3a)^3(-b)^3 + \\ &{}_6C_4(3a)^2(-b)^4 + {}_6C_5(3a)^1(-b)^5 + \\ &{}_6C_6(3a)^0(-b)^6\end{aligned}$$

Evaluate the coefficients in front of each term either by using the formula ${}_nC_r = \frac{n!}{r!(n-r)!}$ or by using the values from row 6 of Pascal's triangle:

$$\begin{aligned}(3a - b)^6 = {}&1(3a)^6 - 6(3a)^5 b + 15(3a)^4 b^2 - \\ &20(3a)^3 b^3 + 15(3a)^2 b^4 - \\ &6(3a)b^5 + 1b^6\end{aligned}$$

Expand the fifth term:

$$15(3a)^2 b^4 = 15 \cdot 9a^2 b^4 = 135a^2 b^4$$

2. **(C)** Use log laws:

$$\begin{aligned}\ln\sqrt[3]{e^2 x} &= \ln(e^2 x)^{\frac{1}{3}}(e^2 x) = \tfrac{1}{3}[\ln e^2 + \ln x] = \tfrac{1}{3} \\ &[2 + \ln x] \\ &= \tfrac{2}{3} + \tfrac{1}{3}\ln x\end{aligned}$$

3. **(B)** Rewrite the expression in terms of sine and cosine:

$$\frac{\cot^2 x}{1+\csc x} = \frac{\frac{\cos^2 x}{\sin^2 x}}{1+\frac{1}{\sin x}} = \frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{\sin x+1}{\sin x}} = \frac{\cos^2 x}{\sin^2 x}\cdot\frac{\sin x}{\sin x+1}$$
$$= \frac{1-\sin^2 x}{\sin^2 x}\cdot\frac{\sin x}{\sin x+1}$$
$$= \frac{(1+\sin x)(1-\sin x)}{\sin^2 x}\cdot\frac{\sin x}{\sin x+1} = \frac{1-\sin x}{\sin x}$$
$$= \frac{1}{\sin x} - \frac{\sin x}{\sin x} = \csc x - 1$$

4. **(D)** Use trigonometric substitutions:

$$\sin\theta\cdot\cos\theta\cdot\tan\theta = \sin\theta\cdot\cos\theta\cdot\frac{\sin\theta}{\cos\theta} = \sin^2\theta$$

Since $\sin\theta = b$, $\sin^2\theta = (\sin\theta)^2 = b^2$.

5. **(A)** Given $f(\theta) = a\cos\theta + b$, the amplitude = a and the midline = b. Therefore, the maximum value is $b + a$ and the minimum value is $b - a$.

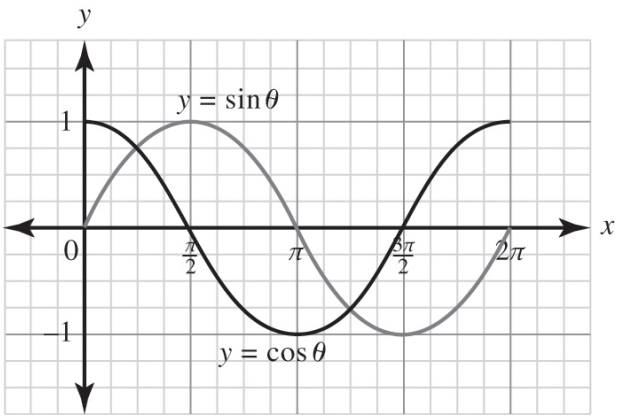
6. **(B)** Since the graph is starting at the origin, the parent function is $y = \sin x$. Since it decreases as x increases from 0, it is a reflection over the x-axis, making the parent function $y = -\sin x$. The amplitude is $2 = a$, and there is half of the curve from $(0, 2\pi)$. So $b = \frac{1}{2}$. Therefore, the transformed equation is $y = -2\sin\left(\frac{1}{2}x\right)$.

7. **(D)** The coterminal angle of $-\frac{\pi}{2}$ is $\frac{3\pi}{2}$ so:

$$\csc\left(-\frac{\pi}{2}\right) = \csc\left(\frac{3\pi}{2}\right) = \frac{1}{\sin\left(\frac{3\pi}{2}\right)} = \frac{1}{-1} = -1$$

The only choice that is equivalent to –1 is $\cos\pi$.

8. **(B)** The graphs of the two functions are shown below. Both are decreasing in the interval $\left(\frac{\pi}{2}, \pi\right)$, which represents Quadrant II.



9. **(B)** Starting with the inner function of the composition, we know that $\tan\theta = \frac{4}{3}$. This creates a right triangle whose side opposite angle θ is 4 and whose side adjacent to angle θ is 3. Using the Pythagorean theorem, the hypotenuse would be 5. Then evaluate:

$$\csc(\theta) = \frac{1}{\sin\theta} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

10. **(A)** Starting with the left side of the equation, $k - \frac{1}{\sec^2 x} = k - \cos^2 x$. For this to equal $\sin^2 x$, then $k = 1$ so that it will match the Pythagorean trigonometric identity $\sin^2 x + \cos^2 x = 1$.

11. **(C)** Use the double-angle trigonometric identities:

$$\frac{\sin 2\theta}{2\tan\theta} = \frac{2\sin\theta\cos\theta}{2\tan\theta}$$

Rewrite $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{2\sin\theta\cos\theta}{2\tan\theta} = \frac{2\sin\theta\cos\theta}{2\frac{\sin\theta}{\cos\theta}} = \frac{2\sin\theta\cos\theta}{2\sin\theta}\cdot\cos\theta$$
$$= \cos^2\theta$$

12. **(A)** Set the equation equal to 0:

$$\sin^2 x - \sin x = 0$$

Factor and set each factor equal to 0:

$$\sin x(\sin x - 1) = 0$$
$$\sin x = 0 \text{ and } \sin x - 1 = 0$$

Isolate sin x and find angle x using the unit circle. For sin $x = 0$, then $x = 0, \pi$. For sin $x = 1$, then $x = \frac{\pi}{2}$.

13. **(C)** Use the identity for sin($A + B$):

$$\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}$$
$$= \sin x(0) + \cos x(1) = \cos x$$

Then:

$$\frac{\sin\left(x + \frac{\pi}{2}\right)}{-\sin x} = \frac{\cos x}{-\sin x} = -\cot x$$

14. **(B)** The parent polar equation for $r = 6 \cos \theta$ can be found in Table 9.2 in Section 9.2. A table of values can also be created that inputs different angle measures for θ from the unit circle, and then the points can be plotted.

15. **(D)** In the polar plane, the point $\left(4, \frac{11\pi}{18}\right)$ lies in Quadrant II with $r = 4$ and $\theta = \frac{11\pi}{18}$. The point in Choice (D) has an angle that is coterminal to $\theta = \frac{11\pi}{18}$ with the same $r = 4$. Therefore, the two points would be in the same location.

16. **(C)** To convert $\left(3, -\frac{3\pi}{4}\right)$ into rectangular coordinates (x, y), use the conversion formulas. Therefore: $r = 3$ and $\theta = -\frac{3\pi}{4}$.

$$x = r\cos\theta = 3\cos\left(-\frac{3\pi}{4}\right) = 3\left(-\cos\left(\frac{5\pi}{4}\right)\right)$$
$$= 3\left(-\cos\left(\frac{\pi}{4}\right)\right) = 3\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$$

Similarly:

$$y = r\sin\theta = 3\sin\left(-\frac{3\pi}{4}\right) = 3\left(-\sin\left(\frac{5\pi}{4}\right)\right)$$
$$= 3\left(-\sin\left(\frac{\pi}{4}\right)\right) = 3\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$$

The equivalent rectangular coordinates are $(x, y) = \left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$.

17. **(D)** $g(f(x)) = g(x^3 - 3x^2 - 2x + 5) = 2$. Since $g(x) = 2$, it is a constant function. This means that no matter the input, the output will always be 2.

18. **(C)** For a function to be odd, graphically, it is symmetric with respect to the origin and algebraically $f(-x) = -f(x)$. For the function h, $h(-x) = (-x)^5 + 1 = -x^5 + 1 \neq -h(x)$ since all the signs were not negated.

19. **(D)** The solutions are the points of intersection of the two graphs, which can be algebraically found by substituting. Substitute $y = x + 2$ into the second equation for y:

$$(x+2)^2 = 4x$$
$$x^2 + 4x + 4 = 4x$$

Set the equation equal to 0 and solve for x:

$$x^2 + 4 = 0$$

The solutions to this equation are imaginary, specifically $x = \pm 2i$. There are no real solutions.

20. **(D)** Given that $\log_b(3^b) = \frac{b}{2}$, using log laws lets the left side of the equation be rewritten as $b \cdot \log_b 3$. Set this equal to the right side of the equation and divide out a common factor of b:

$$b \cdot \log_b 3 = \frac{b}{2}$$
$$\log_b 3 = \frac{1}{2}$$

Rewrite in exponential form:

$$b^{\frac{1}{2}} = 3$$

Square both sides of the equation to solve for b:

$$b = 3^2 = 9$$

21. **(A)** To find the inverse equation, switch x and y and solve for y:

$$x = 2e^{-y}$$
$$\frac{x}{2} = e^{-y}$$

Rewrite as a natural log equation:

$$-y = \ln\left(\frac{x}{2}\right)$$
$$y = -\ln\left(\frac{x}{2}\right)$$

Therefore, the inverse equation is $f^{-1}(x) = -\ln\left(\frac{x}{2}\right)$. Using log laws this is equivalent to:

$$f^{-1}(x) = -\ln\left(\frac{x}{2}\right) = \ln\left(\frac{x}{2}\right)^{-1} = \ln\left(\frac{2}{x}\right)$$

22. **(D)** If $f(x) = 2x^2 + 4x - 1$:

$$\begin{aligned} f(x+h) &= 2(x+h)^2 + 4(x+h) - 1 \\ &= 2(x^2 + 2xh + h^2) + 4(x+h) - 1 \\ &= 2x^2 + 4xh + 2h^2 + 4x + 4h - 1 \end{aligned}$$

23. **(C)** To evaluate $f(6)$, first identify which of the functions to use in the piecewise function. Since $x = 6$, it fits the interval of the second equation. So $f(6) = 4(6) = 24$.

24. **(B)** Starting at the left of the graph, it decreases from $(-\infty, -1)$ then increases from $(-1, 0)$ then decreases from $(0, 1)$ and then increases from $(1, \infty)$.

25. **(C)** The composition $(f \circ g)(4) = f(g(4)) = f(7) = 14$.

26. **(B)** To find the horizontal asymptote, examine the degree of the polynomial in both the numerator and the denominator. Since both are degree 2, the horizontal asymptote is the ratio of the leading coefficients, $y = \frac{7}{3}$.

27. **(A)** If x = length and y = width, the perimeter of the rectangular garden is $1{,}400 = 2x + 2y$. Therefore, the width is $y = 700 - x$. Since the area of a rectangle is the product of the length and the width, $A(x) = x(700 - x) = 700x - x^2$. Since no one side can be all the fencing, the domain must be restricted to $(0, 700)$ so that there is enough fencing for the garden to exist with all four sides.

28. **(C)** Since it is given that $x = -2$ is a real zero of $f(x)$, use long division to divide the factor $(x + 2)$ into $f(x)$. An equivalent representation of $f(x) = (x + 2)(x^2 + 10x + 26)$ after long division. Since the second factor is a quadratic, the zeros can be solved for by using the quadratic formula:

$$x = \frac{-(10) \pm \sqrt{(10)^2 - 4(1)(26)}}{2(1)} = \frac{-10 \pm \sqrt{-4}}{2}$$
$$= \frac{-10 \pm 2i}{2} = -5 \pm i$$

The imaginary solutions are in conjugate pairs and can be written as factors:

$$(x - (-5 - i))(x - (-5 + i)) = (x + 5 + i)(x + 5 - i)$$

So:

$$f(x) = (x + 2)(x + 5 + i)(x + 5 - i)$$

Part B

29. **(D)** Since there are 4 quarters in each year, $n = 4 \cdot 5 = 20$. Substitute $n = 20$ into the sequence:

$$a_{20} = 12{,}000\left(1 + \frac{0.076}{4}\right)^{(20)} = \$17{,}484.97$$

30. **(A)** Since both log expressions are of different bases that do not have a relationship to each other, use the change of base formula to rewrite each of the logarithmic expressions:

$$\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$$
$$\frac{\log(x - 1)}{\log 0.3} < \frac{\log(x - 1)}{\log 0.09}$$

Since log 0.3 > log 0.09, for this inequality to be true then both fractions must be negative, meaning that log(x − 1) must be negative.

This leads to the following inequality:

$$\log(x - 1) < 0$$

Change the inequality to an equal sign and rewrite in exponential form:

$$10^0 = x - 1$$
$$x - 1 = 1$$
$$x = 2$$

To determine if $x < 2$ or $x > 2$, choose x-values in those intervals and test those values in the original inequality.

$x = 1.5$	$x = 3$
$\log_{0.3}((1.5) - 1) < \log_{0.09}((1.5) - 1)$ $0.5757 < 0.287$	$\log_{0.3}((3) - 1) < \log_{0.09}((3) - 1) -$ $0.5757 < -0.287$
This is not a true statement, so x cannot be less than 2.	This is a true statement, so x is greater than 2.

For the final solution, it is important to consider the domain of a logarithmic function, which is all input values greater than 0. If $x - 1 > 0$, then $x > 1$.

Therefore, the final solution is $x > 2$.

31. **(B)** Point $P(-0.8, 0.6)$ lies in Quadrant II. Since P lies on the unit circle, the y-coordinate is $0.6 = \sin\theta$. The reference angle is $\sin -1(0.6) \approx 0.6435011$ radians. The angle in Quadrant II is $\theta \approx \pi - 0.6435011 \approx 2.4981$, which is 2.50 when rounded to the nearest hundredth.

32. **(C)** Finding the minimum value of $C(x) = 2x^2 - 22x + 760$ can be done graphically using technology. It can also be done algebraically by finding the axis of symmetry, $x = -\frac{b}{2a}$, and substituting that x-value into the function $C(x)$:

$$x = -\frac{-22}{2(2)} = 5.5$$
$$C(5.5) = 699.50 \approx 700$$

33. **(D)** Calculate the average rate of change:

$$\frac{\Delta y}{\Delta x} = \frac{h(2) - h(0)}{2 - 0} = \frac{22.4 - 2}{2 - 0} = 10.2$$

34. **(D)** To solve the rational inequality, first get it in the form $f(x) > 0$ by subtracting the 5 from both sides and combining the expressions:

$$\frac{x+2}{2x-1} - 5 > 0$$

$$\frac{x+2-5(2x-1)}{5(2x-1)} > 0$$

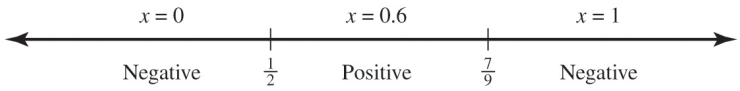
$$\frac{x+2-10x+5}{5(2x-1)} > 0$$

$$\frac{7-9x}{5(2x-1)} > 0$$

Find the critical values by setting the numerator and denominator equal to 0 and solving for x:

$$7 - 9x = 0 \qquad 5(2x-1) = 0$$
$$x = \frac{7}{9} \qquad x = \frac{1}{2}$$

Place these critical values on a number line, and choose test points within each interval to find the sign of $f(x)$.



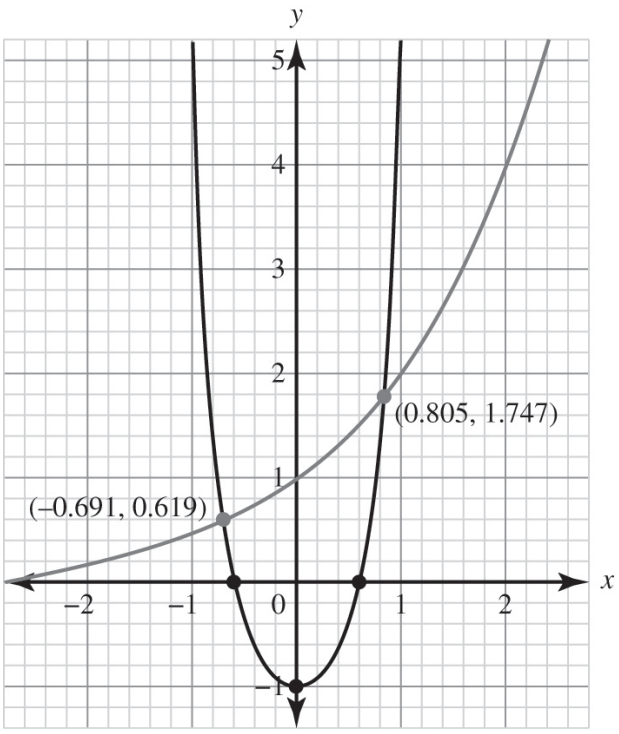
The function $f(x) > 0$ in the interval where the values are positive. Since $f(x) \neq 0$, the endpoints of the intervals are contained by parentheses. The solution is $\left(\frac{1}{2}, \frac{7}{9}\right)$.

35. **(D)** Since the relation is said to be continuous, there are no breaks in the output values. Therefore, a y-value of 0 must exist in an interval where the y-values switch from positive to negative or from negative to positive. This happens in the intervals $(-3, -1) \cup (5, 7)$.

36. **(A)** You are given the function $f(x) = \sqrt{x-a} - \frac{1}{\sqrt{b-x}}, 0 < a < b$. To find the domain, there are two different sets of restrictions. The first expression of the function is a radical expression where the radicand must be greater than or equal to 0. Therefore, $x - a \geq 0 \rightarrow x \geq a$. For the second expression, the function is a rational expression where the denominator has a radical expression. Therefore, $b - x > 0 \rightarrow b > x \rightarrow x < b$. Since it was given that $0 < a < b$, the domain of $f(x)$ is $a \leq x < b$.

37. **(D)** To find the solutions of the equation $5x^4 + x^2 - 1 = 2x$, use technology to graph the two different sides of the equation and find the

points of intersection.



The largest solution is $x = 0.805$, and the smallest solution is $x = -0.691$. The difference between the largest and smallest solutions is $0.805 - (-0.691) = 1.496 \approx 1.50$.

38. **(C)** Looking at the growth of the population, the data resemble an exponential pattern. Using technology, enter the data into the lists of a graphing calculator where L1 = number of years since 1980 and L2 = population (in thousands). Calculate the exponential regression model:

$$y = 1.975896573\,(1.0737245)^x$$

Substitute $x = 44$ for the year 2024:

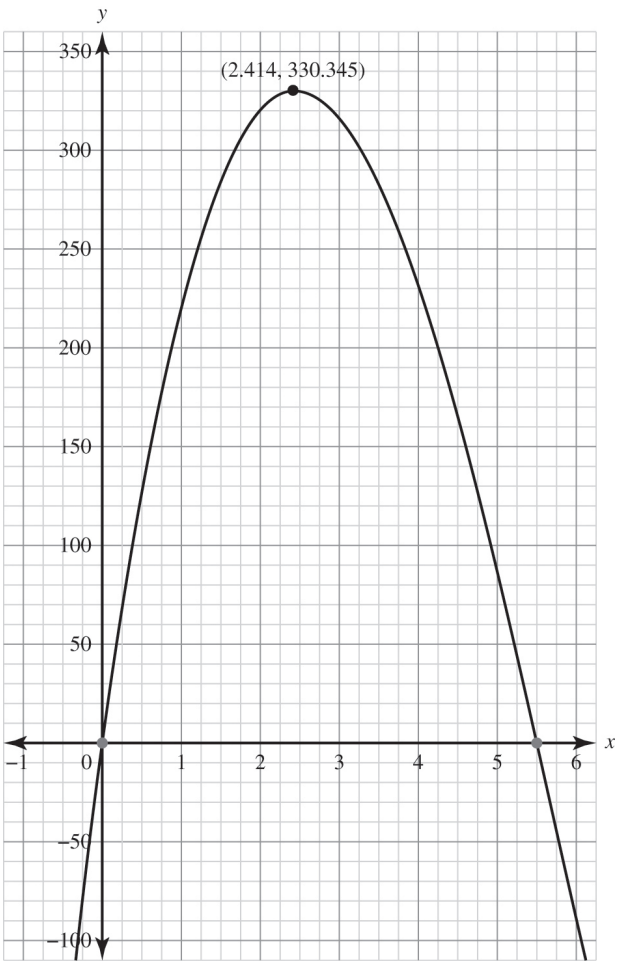
$$\begin{aligned} y &= 1.975896573\,(1.0737245)^{(44)} \\ &= 45.19081499 \approx 45{,}000 \text{ people} \end{aligned}$$

39. **(A)** Let x = length of the side of the square. This will also represent the height of the box. Since 2 squares are being cut on either side of the original piece of cardboard, the length of the box is $27 - 2x$ and the

width of the box is $11 - 2x$. It is necessary to restrict the domain here. Since we want the box to exist, $x > 0$. The width of the box needs to be restricted such that:

$$\begin{aligned} 11 - 2x &> 0 \\ x &< 5.5 \end{aligned}$$

The domain is $0 < x < 5.5$. A function to represent the volume is $V(x) = (27 - 2x)(11 - 2x)(x)$. Using technology, this function can be graphed within the restricted domain and the calculator can determine the coordinates of the maximum.



Then $x = 2.414$, $V = 330.345$. This means that there will be a maximum volume of 330.345 cubic inches when the length of the squares cut out are $2.414 \approx^{2} .41$ inches.

40. **(D)** To convert $(4, -4\sqrt{3})$ into polar coordinates (r, θ), find θ and r when $x = 4$ and $y = -4\sqrt{3}$.

To find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{(4)^2 + (-4\sqrt{3})^2} = \sqrt{16 + (16 \cdot 3)} = \sqrt{16 + 48} = \sqrt{64} = 8$$

To find θ:

$$\theta = \arctan\left(\left|\frac{y}{x}\right|\right) \text{ for } x > 0,\ y < 0$$

$$\theta = -\arctan\left|\frac{-4\sqrt{3}}{4}\right| = -\arctan\left(\frac{4\sqrt{3}}{4}\right)$$

$$= -\arctan(\sqrt{3}) = -\frac{\pi}{3}$$

Find the coterminal angle:

$$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

The equivalent polar coordinates are $(r, \theta) = \left(8, \frac{5\pi}{3}\right)$.

Section II

Part A

1. (A)(i)

$$h(4) = g(f(4))$$

Look at the graph of $f(x)$, and find the point on the function at which the x-value is 4. That point has coordinates (4, 3); therefore, $f(4) = 3$. Then $h(4) = g(f(4)) = g(3)$. Using the equation for $g(x)$, $g(3) = 5.212(6) - 3 \approx 0.024$.

(ii) The zeros for $f(x)$ are defined to be the x-values for which the function is equal to 0. Graphically, this occurs at the x-intercepts. Therefore, the zeros are $x = 0, 1.5, 2.7, 7$.

(B)(i) Set the $g(x)$ equation equal to –2:

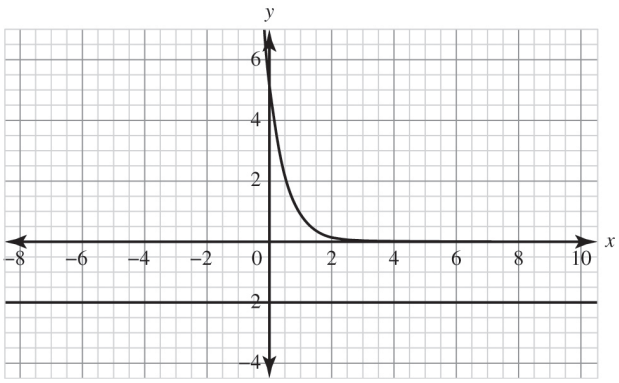
$$5.212(6)^{-x} = -2$$

Isolate the exponential term by dividing both sides of the equation by 5.212:

$$(6)^{-x} = \frac{-2}{5.212}$$

To solve for x, take the logarithm of both sides of the equation. However, since the domain of a logarithmic function is $x > 0$, it is not possible to take the logarithm of $\frac{-2}{5.212}$. Therefore, there is no solution.

This can also be confirmed by graphing the two sides of the equation, using a graphing calculator, and noting that the graphs, as shown below, do not intersect.



(ii) As seen on the graph of $g(x)$, as x increases without bound, $g(x)$ follows along the x-axis, approaching 0. Therefore:

$$\lim_{x \to \infty} g(x) = 0$$

(C)(i) There is no inverse function for $f(x)$.

(ii) To determine graphically if f has an inverse function, use the horizontal line test. Since there are places where a horizontal line would intersect the graph in more than one point, there is no inverse function for $f(x)$.

2. (A)(i) Substitute the coordinate pairs (0, 47.3) and (110, 78.7) into the equation $L(t) = a + b \ln (t + 1)$:

$$47.3 = a + b\ln(0 + 1)$$
$$78.7 = a + b\ln(110 + 1)$$

(ii) To solve for *a* and *b*, simplify the first equation by substituting 0 for ln(1):

$$47.3 = a + b(0)$$
$$47.3 = a$$

Substitute 47.3 for *a* into the second equation to solve for *b*:

$$78.7 = 47.3 + b\ln(110 + 1)$$
$$78.7 = 47.3 + b\ln(111)$$

Solve for *b* by subtracting 47.3 and dividing both sides by ln (111):

$$31.4 = b\ln(111)$$
$$b = \frac{31.4}{\ln(111)} \approx 6.667$$

(B)(i) Average rate of $= \frac{78.7 - 47.3}{110 - 0} = \frac{31.4}{110} \approx 0.285$

(ii) The average life expectancy increases approximately 0.285 years each year.

(iii) To find the average life expectancy in 1950, write a linear approximation equation using the given ordered pair (0, 47.3) and the average rate of change, 0.285:

$$y - 47.3 = 0.285(x - 0)$$

Substitute 50 for *x* and solve for *y*:

$$y - 47.3 = (0.285)(50 - 0) \rightarrow y = 61.55$$

The average life expectancy in 1950 was approximately 61.55 years.

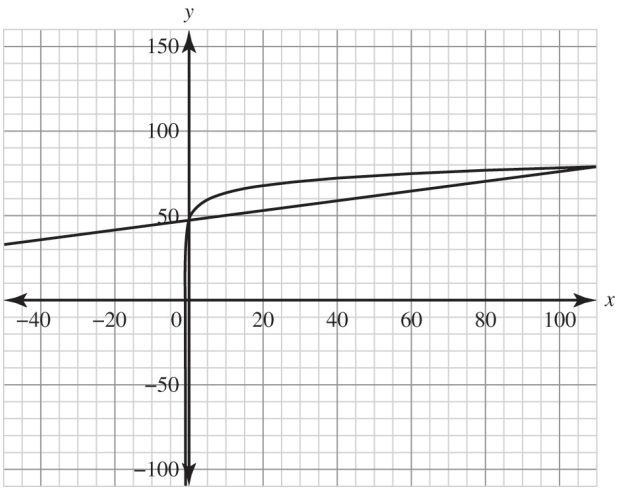
(C)(i) The logarithmic model is

$L(t) = 47.3 + 6.667 \ln(t + 1)$. Therefore:

$$L(50) = 47.3 + 6.667\ln(51) \approx 73.513 \text{ years}$$

(ii) The value found with the linear approximation is less than the value found from the logarithmic model. The logarithmic model is increasing

at a decreasing rate since the graph of $L(t)$ is concave down. On the given interval, the average rate of change of the secant line (the graph of the linear approximation) lies below the graph of $L(t)$. This can be verified by the graphs of $L(t)$ from (C) and the secant line from (B) shown below.



Part B

3. (A) Using the given information, create corresponding height values above the ground with times. Initially, at $t = 0$, the ball is at a height of 1.8 m, which could correspond to the coordinates (0, 1.8). This can represent point *E*. Then $\frac{3}{2}$ seconds later, the ball travels an additional 0.4 m upwards to point *P* and then back down 0.4 m to return to point *C*. This would lead to the coordinates $\left(\frac{3}{4}, 2.2\right)$ and $\left(\frac{3}{2}, 1.8\right)$, which can correspond to points *H* and *K*, respectively. The ball continues following this pattern traveling from point *C* to point *P* and returning to point *C*.

   The following information can be concluded: the maximum height is 2.2 m, the minimum height is 1.8 m, and the time it takes to complete one full cycle is $\frac{3}{2}$ seconds. Midway between the minimum height and maximum height is $\frac{1.8 + 2.2}{2} = \frac{4}{2} = 2$. This represents the *y*-coordinates of points *G* and *J*. To find the *x*-coordinates, the period can be divided by $4: \frac{3/2}{4} = \frac{3}{8}$. Thus, point *G* can correspond to $\left(\frac{3}{8}, 2\right)$ and point *J* can correspond to $\left(\frac{9}{8}, 2\right)$.

(B) For the function h:

- a represents the amplitude or distance from the midline to the maximum value: $a = \frac{2.2 - 1.8}{2} = 0.2$
- b represents the number of full cycles from 0 to 2π: $b = \frac{2\pi}{\frac{3}{2} - 0} = \frac{4\pi}{3}$
- c represents the horizontal shift right from the origin: $c = -\frac{3}{8}$
- d represents the vertical shift up from the origin: $d = 2$

Therefore, $h(t) = 0.2 \sin\left(\frac{4\pi}{3}\left(t - \frac{3}{8}\right)\right) + 2$.

(C)(i) h is positive and decreasing on the interval (t_3, t_5).

(ii) Since the graph of h is concave down on the interval (t_3, t_4), the rate of change of h is decreasing on the interval (t_3, t_4).

4. (A)(i) Use log laws to condense:

$$f(x) = \log_{16}(x^2 - x + 4) - \log_{16}(2x + 11)$$
$$f(x) = \log_{16}\frac{x^2 - x + 4}{2x + 11}$$

(ii) Set the equation equal to $\frac{3}{4}$:

$$\frac{3}{4} = \log_{16}\frac{x^2 - x + 4}{2x + 11}$$

Rewrite in exponential form:

$$16^{\frac{3}{4}} = \frac{x^2 - x + 4}{2x + 11}$$
$$8 = \frac{x^2 - x + 4}{2x + 11}$$

Multiply both sides of the equation by $2x + 11$:

$$8(2x + 11) = x^2 - x + 4$$

Use the distributive property:

$$16x + 88 = x^2 - x + 4$$

Rewrite the quadratic equation in standard form:

$$x^2 - 17x - 84 = 0$$

Factor:

$$(x - 21)(x + 4) = 0$$

Solve for x by setting each factor equal to 0:

$$x = 21,\ x = -4$$

Check for extraneous solutions by substituting each value into the original equation:

$x = 21$	$x = -4$
$\frac{3}{4} = \log_{16}\frac{(21)^2 - (21) + 4}{2(21) + 11}$	$\frac{3}{4} = \log_{16}\frac{(-4)^2 - (-4) + 4}{2(-4) + 11}$
$\frac{3}{4} = \log_{16} 8$	$\frac{3}{4} = \log_{16} 8$
$\frac{3}{4} = \frac{3}{4}$	$\frac{3}{4} = \frac{3}{4}$

Both solutions check with the original equation and therefore the solutions are $x = 21$ and $x = -4$.

(iii) Set the equation equal to –66:

$$-66 = -10e^{2-2x} - 6$$

Isolate the exponential term by adding 6 and dividing by –10 on both sides of the equation:

$$6 = e^{2-2x}$$

Rewrite in log form:

$$\ln 6 = 2 - 2x$$

Solve for x by subtracting 2 and dividing by –2 on both sides of the equation:

$$x = \frac{\ln(6) - 2}{-2}$$

(B)(i) Use a double-angle identity to substitute for cos(2*x*):

$$k(x) = \cos(2x) - \sin(x)$$
$$k(x) = 1 - 2\sin^2 x - \sin(x)$$

Rewrite in standard quadratic form:

$$k(x) = -(2\sin^2 x + \sin x - 1)$$

Factor:

$$k(x) = -(2\sin^2 x + 2\sin x - \sin x - 1)$$
$$k(x) = -[2\sin x(\sin x + 1) - 1(\sin x + 1)]$$
$$k(x) = -(2\sin x - 1)(\sin x + 1)$$

(ii) Set the factored equation equal to 0:

$$0 = -(2\sin x - 1)(\sin x + 1)$$

Set each factor equal to 0 and isolate sin *x*:

$$-(2\sin x - 1) = 0 \qquad (\sin x + 1) = 0$$
$$\sin x = \frac{1}{2} \qquad \sin x = -1$$

Use the unit circle to solve for *x*:

For $\sin x = \frac{1}{2}, x = \frac{\pi}{6}$ and $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

For $\sin x = -1, x = \frac{3\pi}{2}$. Since this is not in the domain, use the cofunction $x = -\frac{\pi}{2}$.

(C) Set the equation equal to $\frac{\sqrt{3}}{3}$:

$$\frac{\sqrt{3}}{3} = \tan\left(\cos^{-1}\left(\frac{\sqrt{3}}{x}\right)\right)$$

Rewrite the equation using inverse tangent:

$$\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{x}\right)$$

Use the unit circle to evaluate $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$:

$$\frac{\pi}{6} = \cos^{-1}\left(\frac{\sqrt{3}}{x}\right)$$

Rewrite the equation using cosine:

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{x}$$
$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{x}$$

Solve for x:

$$x = 2$$

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Glossary of Key Terms

Amplitude Half the difference between the maximum and minimum values of a sinusoidal function.

Arithmetic sequence A sequence in which consecutive terms have a common difference, d, or a constant rate of change.

Asymptote A line whose distance from a curve approaches 0 as the curve tends to infinity.

Complex zero For a function, $f(x)$, a complex zero is a value of x that makes $f(x) = 0$ and is of the form $a \pm bi$, $b \neq 0$.

Composition Links two functions to form a new function by using the output of one function as the input to the other function.

Concavity Describes the curvature of the graph of a function; a function is concave up if the rate of change is increasing; a function is concave down if the rate of change is decreasing.

Cosecant function Notated as $f(\theta) = \csc \theta$; it is the reciprocal of the sine function, where $\sin \theta \neq 0$.

Cotangent function Notated as $f(\theta) = \cot \theta$; it is the reciprocal of the tangent function, where $\tan \theta \neq 0$.

Decreasing A function with a domain and range of real numbers such that the dependent variable decreases as the independent variable increases.

Dependent variable The variable representing output values; typically notated using the letter y.

Determinant The determinant of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$ and is notated as det(A).

Domain The set of values of a function that can be assumed by the independent variable of the function; typically notated using the letter x.

Dot product The dot product of two vectors is the sum of the products of their corresponding components; that is, $\langle a_1, b_1 \rangle \cdot \langle a_2, b_2 \rangle = a_1 a_2 + b_1 b_2$.

Error The difference between the predicted and actual values of a model.

Exponential function A function of the general form $f(x) = ab^x$, with the initial value a, where $a \neq 0$, and the base b, where $b > 0$ and $b \neq 1$; when $a > 0$ and $b > 1$, the exponential function is said to demonstrate exponential growth; when $a > 0$ and $0 < b < 1$, the exponential function is said to demonstrate exponential decay.

Factor A number or polynomial that divides a given number or polynomial exactly; for example, 1, 2, 3, and 6 are all factors of 6; $x - 1$ and $x + 2$ are factors of $x^2 + x - 2$ since $(x - 1)(x + 2) = x^2 + x - 2$.

Function A mathematical relation that maps a set of input values to a set of output values such that each input value is mapped to exactly one output value.

Geometric sequence A sequence in which consecutive terms have a common ratio, r, or a constant proportional change.

Holes Exist if there is a common factor in the numerator and denominator of a rational function that can be removed and then graphed using an open circle; holes are also known as removable discontinuities.

Horizontal asymptote Occurs on the graph of a rational function when the degree of the numerator is less than or equal to the degree of the

denominator.

Identity function $f(x) = x$.

Identity matrix A square matrix consisting of 1's on the diagonal from the top left to bottom right and 0's everywhere else; notated as I.

Increasing A function with a domain and range of real numbers such that the dependent variable increases as the independent variable increases.

Independent variable The variable representing input values; typically notated using the letter x.

Inverse function A reverse mapping of a function; an inverse function, f^{-1}, maps the output values of a function, f, on its invertible domain to their corresponding input values; that is, if $f(a) = b$, then $f^{-1}(b) = a$.

Limit A value of a function that can be approached arbitrarily closely by the dependent variable when some restriction is placed on the independent variable of the function.

Linear function May be represented as an equation that has the form $y = ax + b$, $a \neq 0$; the graph of a linear function is a line with a constant slope.

Linear transformation A function that maps an input vector to an output vector such that each component of the output vector is the sum of constant multiples of the input vector components.

Logarithmic expression Notated as $\log_b c$; is equal to, or represents, the value that the base b must be exponentially raised to in order to obtain the value c; that is, $\log_b c = a$ if and only if $b^a = c$, where a and c are constants, $b > 0$, and $b \neq 1$.

Logarithmic function A function of the general form $f(x) = a \log_b x$, with base b, where $b > 0$, $b \neq 1$, and $a \neq 0$.

Magnitude The magnitude of a vector is the length of the line segment of the vector a, b , which is found by evaluating $\sqrt{a^2 + b^2}$.

Matrix A set of quantities (called elements or entries) arranged in a rectangular array with certain rules governing their combination; the array is enclosed in square brackets; the horizontal lines of elements are rows, and the vertical lines are columns.

Midline Determined by the average of the maximum and minimum values of a sinusoidal function.

Multiplicity Refers to a multiple root or a repeated root of an equation; in general, if $(x - r)^n$ is a factor of a polynomial equation, then r is a root of multiplicity n of the equation.

Parabola A U-shaped curve that is the graph of a quadratic function; the slope of a parabola changes along the curve.

Parametric function In $\mathbb{R}^2$, the set of all ordered pairs of two real numbers, consists of a set of two parametric equations in which two dependent variables, x and y, are dependent on a single independent variable, t, called the parameter.

Period The period of a function can be defined as the smallest positive value k such that $f(x + k) = f(x)$ for all x in the domain.

Periodic relationship Identified between two aspects of a context if, as the input values increase, the output values demonstrate a repeating pattern over successive equal-length intervals.

Piecewise-defined function Consists of a set of functions defined over nonoverlapping domain intervals.

Polar coordinates An ordered pair, (r, θ), such that $|r|$ represents the radius of the circle on which the point lies and θ represents the measure of an angle in standard position whose terminal ray includes the point.

Polynomial function A function that can be written in the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n is a positive integer called the degree of the polynomial and $a_n \neq 0$.

Quadratic function A second-degree equation that has the form $y = ax^2 + bx + c$, $a \neq 0$.

Radian measure The radian measure of an angle in standard position is the ratio of the length of the arc of a circle centered at the origin subtended by the angle to the radius of that same circle.

Range The set of output values of a function.

Rate of change A comparison that describes how one quantity changes in relation to another quantity; rates of change can be either positive or negative.

Rational function A function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$; graphs of rational functions may have discontinuities, such as asymptotes or "holes."

Real zero For a function, $f(x)$, a real zero is a real value of x that makes $f(x) = 0$.

Regression A model that describes the dependence of the mean value of one random variable on one or more other variables.

Residuals In statistics, residuals are the differences between observed values and values predicted by a model; residuals are sometimes called errors.

Secant function Notated as $f(\theta) = \sec \theta$; it is the reciprocal of the cosine function, where $\cos \theta \neq 0$.

Semi-log plot A plot that has one of the axes logarithmically scaled; when the y-axis of a semi-log plot is logarithmically scaled, data or functions that

demonstrate exponential characteristics will appear linear.

Sequence A function from the whole numbers to the real numbers; the graph of a sequence consists of discrete points instead of a curve.

Sinusoidal function Any function that involves additive and multiplicative transformations of $f(\theta) = \sin\theta$; the sine and cosine functions are both sinusoidal functions with $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$.

Standard position The standard position of an angle in the coordinate plane occurs when the vertex coincides with the origin and one ray coincides with the positive *x*-axis; the other ray is called the terminal ray.

Tangent function Notated as $f(\theta) = \tan\theta$; it is the ratio of the sine function and the cosine function, where $\cos\theta \neq 0$.

Transformation Transformation of a function moves each point of the graph according to a given rule such as a reflection, translation, or dilation.

Unit circle A circle that is centered at the point (0, 0) with a radius length of 1.

Unit vector A vector of magnitude 1.

Vector A directed line segment; when a vector is placed in the plane, the point at the beginning of the line segment is called the tail, and the point at the end of the line segment is called the head.

Vertical asymptote Occurs on the graph of a rational function at a value of *x* for which the denominator evaluates to 0 but the numerator does not evaluate to 0; it is written as an equation of a vertical line $x = a$ and is typically graphed as a dashed line.

***x*-intercept** The coordinates of the intersection of a graph with the *x*-axis.

***y*-intercept** The coordinates of the intersection of a graph with the *y*-axis.

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